Reading: Carroll & Ostlie sections 10.3-10.6

Exercises [125 pts total]:

(1) Narrowing in on the atmospheric properties of the Sun [30 pt]

(a) [5 pts] Show that the equation of hydrostatic equilibrium can be combined with the definition for optical depth to derive:

\[ \frac{dP}{d\tau} = \frac{g}{\kappa} \]

(b) [5 pts] Calculate the pressure of the Sun’s photosphere (\(\tau = 2/3\)) estimating \(\kappa = 0.1 \text{ m}^2/\text{kg}\).

(c) [5 pts] Assume that the photosphere is a mostly neutral ideal gas with \(X = 0.70, Y = 0.28\) and \(Z = 0.02\). Compute the density at the photosphere for \(T_{\text{phot}} = T_{\text{eff}} = 5800 \text{ K}\). Note: you will need to first derive the proper mean molecular weight \(\mu\).

(d) [5 pts] The primary source of opacity in the Sun’s (and other cool stars’) atmosphere is H- opacity, which arises when a neutral Hydrogen atom becomes slight polarized in the presence of a free electron, resulting in a weak \((\chi = 0.754 \text{ eV})\) bound state. Assuming that this is the sole source of opacity, calculate \(\kappa H\) from equation 9.28 for the conditions derived above. Was the initial estimate for \(\kappa\) correct?

(e) [10 pts] Repeat steps a-d until you converge on values for \(P, \rho\) and \(\kappa\) for the Solar photosphere. “Converge” in this case means that the last iteration differed from the previous iteration by < 10%. This is a common method for determining model parameters, and is most easily done with a computer!
(2) Internal properties of a solid star [25 pt]

Consider a star with mass $M$, radius $R$ and uniform density,

$$\rho_0 = \frac{3}{4\pi} \frac{M}{R^3}$$

(a) [10 pts] Using the equation of hydrostatic equilibrium, show that the pressure as a function of radius in this star can be written as:

$$P(r) = \frac{2\pi}{3} G \rho_0^2 (R^2 - r^2)$$

(b) [10 pts] Show that the core pressure of this star can be written in the form

$$P_c = k G \frac{M^2}{R^4}$$

and find the value of the constant $k$.

(c) [5 pts] Using the solar values for $M$ and $R$, determine numerical values for $\rho_0$ and $P_c$. Also determine the core temperature, $T_c$, assuming an ideal, fully ionized gas with $X = 0.00$, $Y = 0.36$, $Z = 0.64$ (Note: this is different than the atmosphere!). Is this sufficient for proton fusion?

(3) Degenerate Matter [40 pt]

Degenerate matter is defined as fermionic matter that has filled all of its available quantum states. Electron-degenerate gas characterizes the interiors of the lowest mass stars, brown dwarfs, planets and white dwarfs. It can be defined by the distribution function:

$$f(E) = \text{constant for } E \leq \varepsilon_F$$

$$= 0 \text{ for } E > \varepsilon_F$$

where $\varepsilon_F$ is the Fermi energy
(a) [10 pts] Show that, in the case of a nonrelativistic electron gas, the momentum volume element can be written in terms of energy:
\[
d^3 p = 2\pi (2m)^{3/2} E^{1/2} dE
\]

(b) [10 pts] Enforcing the normalization
\[
\int f(E) d^3 p d^3 x = N
\]
and using the conversion from (a), show that the constant for the distribution is equal to \( \frac{3}{4} \pi n \left( \frac{3}{8\pi} (2m\epsilon) \right)^{3/2} n \), where \( n = N/V \) is the particle number density. Don't forget about degeneracy!

(c) [15 pts] Using this distribution, find the average energy \( \langle E \rangle \) per particle, average velocity \( \langle v \rangle \) per particle, and pressure \( \langle P \rangle \) (remember our derivation of \( \langle P \rangle \) for an ideal gas in class).

(d) [5 pts] As derived in section 16.3, the Fermi energy is
\[
\epsilon_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 n}{2} \right)^{2/3}
\]
Use this to show that the pressure of a nonrelativistic degenerate gas scales as \( \rho^{5/3} \).

(4) Some useful polytrope relations [30 pt]

(a) [10 pts] Show that for an \( n=1.5 \) polytrope, the combination \( MR^3 \) is a constant.

(b) [10 pts] Show that for an \( n=3 \) polytrope, the total mass of a star is independent of the central density.

(c) [10 pts] Show that for an \( n=5 \) polytrope, even though \( \xi_i \to \infty \), the total mass remains finite.