Physics 160
Stellar Astrophysics
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Lecture 18
Star Evolution IV: Degenerate Matter and Chandrasekhar Mass Limit
3 December 2013
Announcements

• Remote observing lab is IFFY for tonight; currently there is high humidity and clouds – email will be sent around

• Project papers deadline -> Friday at 4pm (email PDF preferred)

• CAPE evaluations = pizza (currently at 5/16)

• Requiem for Comet ISON... http://www.youtube.com/watch?v=kcROVqmf9SY&feature=youtu.be

• Physics seminar this week (4pm Thursday) – when you want to find dark matter, go to a dark place
**Large Underground Xenon experiment**

250 kg (1/4 ton) of Xenon in a water-enclosed tank 1 mile underground in South Dakota

Able to detect the interaction cross section of a dark matter WIMP to *zeptobarn* scales ($10^{-49} \text{ m}^2$)
Lecture 18: Star Evolution IV: Degenerate Matter and the Chandrasekhar Limit
2 December 2013

PRELIMS
• Announcements [5 min]

MATERIAL [70 min]
• [15 min] Review
• [20 min] Stellar end states
• [10 min] White dwarfs
• [25 min] Degenerate matter

DEMONSTRATIONS/EXERCISES

MATERIALS
Review
• low mass stars ($M < 4 \, M_{\text{Sun}}$) evolve through to AGB, planetary nebula, and finally white dwarf stage (lowest-mass remnant)
• composition – primarily CNO, with He/H atmosphere
• extremely high density – degenerate matter

Degenerate matter
• Motivate: ideal gas vs. electrons in atom – the latter are more constrained
• Consider the case of zero temperature gas – classically all at $E = 0$, but these are Fermi particles => fill states up
• Derive Fermi energy
• Derivation of degeneracy condition: $E_{\text{ideal}} < E_{\text{fermi}}$
• Case studies: sun, white dwarf
• Scaling relation: degeneracy important for $M < 0.3 \, M_{\text{Sun}}$
• Degeneracy pressure - nonrel
  o $n = 1.5$ polytrope
  o Mass-radius relationship

Chandrasekhar limit – relativistic degeneracy
• Relativistic limit: $\rho \approx 10^9 \, \text{kg/m}^3$
• Relativistic pressure < nonrelativistic pressure – star has less support (smaller)
• Derive relativisitc pressure – $n=3$ polytrope
• Limit: $P_{\text{degeneracy}} < P_{\text{grav}} => M_{\text{limit}} = 1.4 \, M_{\text{Sun}}$
• How does this happen? Mass donation by massive companion => novae
• Outcome – if “exposed” WD, carbon detonation => Type Ia SN
Degenerate Matter

In degenerate matter, particles occupy all of the lowest energy states available; this is different from a Maxwell-Boltzmann distribution.

\[ n(E) = \frac{2}{\sqrt{\pi}} \left( \frac{n}{uT} \right)^{3/2} E^{1/2} e^{-E/uT} \]

\[ n(E) \]

\[ E/uT \]

\[ \text{Peak} \sim 0.5 \]

\[ E/F \]

\[ E/F / uT \]

\[ n(E) \]

For "free particles", density of possible states determined by quantum statistics and specifically overlap in deBroglie wavelengths:

For radiation (e.g. waveguide) these correspond to allowable radiative transmission modes:

For quantum particles, \( L = d \rho \Rightarrow h/\rho \Rightarrow p_x^2 = \frac{h^2}{2L} \)
For a non-relativistic particle, \( E = \frac{p^2}{2m} \)

\[
E = \frac{1}{2m} \left( px + py + pz \right)^2
\]

\[
E = \frac{\hbar^2}{8m L^2} \left( n_x^2 + n_y^2 + n_z^2 \right)
\]

\[
E = \frac{\hbar^2 n^2}{8m L^2}
\]

If geometry is spherically symmetric, and \( n \geq 0 \), then the number of possible states is just the volume of a spherical quadrant.

\[
N_{\text{states}} = \frac{1}{8} \left( \frac{4\pi}{3} n_{\text{max}}^3 \right) \times 2
\]

2 spin states per energy state.

Degeneracy for fermions.

Density of states:

\[
\frac{N_{\text{states}}}{L^3} = \frac{\pi}{3} \left( \frac{8m E_{\text{max}}}{\hbar^2} \right)^{3/2}
\]

\[
\rho = \frac{N}{V} = \text{density of particles}
\]

\[
E_{\text{max}} = E_F = \frac{n^{2/3}}{3} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{8m}
\]

\[
E_F = \left( \frac{3\pi^2 n}{2} \right)^{2/3} \frac{\hbar^2}{2m}
\]

\( E_F \) is highest when \( n \) is small \( \Rightarrow \) electrons provide strongest Fermi pressure (if present!)
Condition for degeneracy

For a non-relativistic electron gas, \[ n_e = \frac{e^2}{\text{nucleon} \times \text{volume}} = \frac{\frac{e^2}{2m_e}}{\frac{\rho}{M_H}} \]

\[ \Rightarrow \quad E_F = \frac{k^2}{2m_e} \left( \frac{3\pi^2 \frac{\rho}{A M_H}}{3} \right)^{\frac{2}{3}} \]

The thermal energy per electron \( \approx \frac{3}{2} kT \). If \( \text{Ethermal} < \text{Efermi} \), particle has no freedom of movement \( \Rightarrow \) degenerate

\[ \Rightarrow \quad \frac{3}{2} kT < \frac{k^2}{2m_e} \left( \frac{3\pi^2 \frac{\rho}{A M_H}}{3} \right)^{\frac{2}{3}} \]

\[ \Rightarrow \quad T/\rho^{\frac{2}{3}} < \frac{k^2}{3m_e} \left( \frac{3\pi^2 \frac{\rho}{A M_H}}{3} \right)^{\frac{2}{3}} \approx 1261 \text{ km}^2/\text{kg}^{\frac{2}{3}} \]

For \( \frac{3}{2} A = \frac{1}{2} \):

\( \left( \frac{3 H_{\text{He}}, 12 C}{10 \text{ eV}} \right) \)

Case Study:

The Sun:

\[ T_C \sim 1.5 \times 10^7 \text{ K} \]

\[ \rho_C \sim 1.5 \times 10^5 \text{ kg/m}^3 \]

\[ \frac{7}{8} \approx 0.45(0.5) + 0.35(1) = 0.675 \]

\[ \Rightarrow \quad T/\rho^{\frac{2}{3}} \approx 5300 > 1261 \times \left( \frac{0.65}{0.5} \right)^{\frac{2}{3}} \Rightarrow \text{NOT DEGENERATE} \]

Sirius B:

\[ T_C \sim 8 \times 10^7 \text{ K} \]

\[ \rho_C \sim 6 < \rho \sim 10^{10} \text{ kg/m}^3 \]

\[ \Rightarrow \quad T/\rho^{\frac{2}{3}} \sim 17 < 1261 \Rightarrow \text{DEGENERATE} \]

Brown Dwarf:

\[ T_C \sim 10^6 \text{ K} \]

\[ \rho_C \sim 6 < \rho \sim 6 \left( \frac{3}{4\pi} \right) \frac{50 \text{ MJ}}{\text{m}^3} \sim 2 \times 10^5 \text{ kg/m}^3 \]

\[ \Rightarrow \quad T/\rho^{\frac{2}{3}} \sim 184 < 1261 \Rightarrow \text{DEGENERACY IMPORTANT} \]
In general:

\[ T_c / R_c \propto \left( \frac{M}{R} \right) \left( \frac{R^3}{M^3} \right)^{2/3} \]

\[ \propto M^{2/3} \propto R \propto M^{0.6} \]

\[ \propto M^{1.1} \]

\[ \Rightarrow T_c / R_c \approx 5300 \left( \frac{M}{M_\odot} \right)^{1.1} < 1261 \]

if \( M \leq 0.3 M_\odot \), degeneracy is important for low mass stars.

**Degeneracy pressure**

From our prior derivation of ideal gas pressure, we found:

\[ p = \frac{1}{3} \int_0^\infty n(p) p v dp \]

\[ n(e) = \text{constant} \quad e < E_F \]

\[ \int_0^{E_F} n(e) de = C_F \quad \Rightarrow \quad C_F = N_e \]

\[ n(p) = n(e) \frac{de}{dp} = \frac{N_e}{E_F} p / m_e \]

\[ p = \frac{1}{3} \int_0^{E_F} \frac{N_e}{E_F} \frac{p}{m_e} p \frac{p}{m_e} \frac{p}{m_e} dp - \frac{1}{3} \frac{N_e}{E_F} \frac{1}{4 m_e^2} \frac{1}{4 m_e^2} \frac{1}{4 m_e^2} \]

\[ p_F = \left( \frac{3 \pi^2}{2} \right)^{1/3} \frac{E_F^2}{m_e} \left( \frac{2 m_e}{A m_e} \right)^{5/3} \]

\[ \Rightarrow \quad p \propto \rho^{5/3} \]

\[ \Rightarrow \quad n \approx 1.5 \text{ polytrope} \]

\[ \Rightarrow \quad R \propto (\rho)^{-1/3} \]

\[ \Rightarrow \quad R \leq M^{-1/3} \]
Chandrasekhar limit

We saw earlier that an isothermal core could not withstand gravitational pressure when it reached a certain mass (set by ratios of core-to-envelope mean molecular masses). There is a similar mass limit for a white dwarf based on relativistic effects.

As densities begin to exceed $10^9$ kg/m$^3$, Fermi electrons begin to become relativistic:

$$\mathcal{V} \sim \left( \frac{2e^2}{m_e} \right)^{1/2} \frac{h}{m_e} \left( \frac{3/2}{\hbar} \frac{\pi}{A} \frac{p}{m_e} \right)^{1/3}$$

$$\mathcal{V} \sim 2 \times 10^8 \text{ m/s}$$

By not accounting for relativistic effects, velocities are too large, and so are pressures $\Rightarrow$ actual star is smaller.

Estimate effects: $V \sim c$

$$\Rightarrow \quad P = \frac{1}{3} \int_0^p \left( \frac{n_e}{e_F} \frac{p^2}{m_e} \right) \rho \, dp$$

$$= \frac{1}{3} \frac{n_e}{e_F} \frac{c}{m_e} \frac{1}{3} \frac{p^3}{p}$$

$$= \frac{2\sqrt{3}}{3} \frac{n_e}{e_F} \frac{c}{m_e} m_e^2 \left( \frac{e_F}{2m_e} \right)^{3/2}$$

$$= \frac{2\sqrt{3}}{3} \frac{n_e}{e_F} \frac{c}{m_e} \frac{m_e^2}{e_F} \left( \frac{e_F}{2m_e} \right)^{1/2}$$

$$= \frac{2\sqrt{3}}{3} \frac{n_e}{e_F} \frac{c}{m_e} \frac{m_e}{e_F} \left( \frac{e_F}{2m_e} \right)^{1/2}$$

$$= \left( \frac{2\sqrt{3}}{3} \right) \frac{n_e c}{e_F} \left( \frac{1}{V_3^2} \right) \left( \frac{3/2}{\hbar} \frac{\pi}{4} \right)^{1/3} \left( \frac{3/2}{\hbar} \frac{\pi}{4} \right)^{1/3}$$

more careful:

$$\frac{2\sqrt{3}}{3} \frac{n_e c}{e_F} \left( \frac{1}{V_3^2} \right) \left( \frac{3/2}{\hbar} \frac{\pi}{4} \right)^{1/3} \left( \frac{3/2}{\hbar} \frac{\pi}{4} \right)^{1/3} \leq P \propto \rho^{4/3} \Rightarrow \text{N=3 polytrope}.$$
This is the maximum pressure in a relativistic degenerate core, and ultimately sets the limit on how massive the core can be.

The gravitational pressure pushing downward is:

$$P_c = \frac{G M^2}{R^4}$$

when $$P_c = \frac{C_2 \rho}{\rho}$$

$$\uparrow$$

where $$C_2 \approx 5.5$$ for

$$n=3$$ polytropic

$$\approx 7.5$$ for $$n=3$$

$$P_{\text{gravitation}} > P_{\text{degeneracy}}$$ when

$$C_1 \frac{GM^2}{R^4} > \frac{31T}{4} \frac{1}{C_1} \left( \frac{\rho}{M/M_H} \right)^{1/3}$$

$$\Rightarrow \quad M_{1/3} > \frac{3}{16\pi} \left( \frac{31T}{4} \right)^{1/3} \frac{C_2^{1/3}}{C_1} \left( \frac{\rho}{M/M_H} \right)^{1/3}$$

$$\Rightarrow \quad M > \sqrt{\frac{3}{2} \cdot \frac{31T}{4} \cdot \frac{C_2^2}{C_1^2} \left( \frac{\rho}{M/M_H} \right)^{1/2}} \left( \frac{\rho}{M/M_H} \right)^{1/2}$$

$$\uparrow$$ quantum, relativistic, and gravitating in one term = Placzek

$$\sqrt{\frac{\rho}{M}} = 2 \times 10^{-8} \text{ kg}$$

For $$\frac{\rho}{M} = 0.5$$, $$C_1 = 7.5$$, $$C_2 = 5.5$$ this all reduces to

$$M > 1.4 M_6$$

while over mass limit.
Chandrasekhar limit for neutron stars

Like white dwarfs, eventually density is too high for neutron degeneracy pressure to support a star. This has less to do with relativistic nature of neutrons, but more with sound speed in material:

\[ V_{\text{sound}} = \sqrt{\frac{8\rho}{\rho}} \approx \sqrt{\frac{(3\pi^2)^{\frac{1}{3}} h^2}{5 m_H \left( \frac{\rho}{m_H} \right)^{\frac{2}{3}}} \rho} \]

\[ \approx 70 \rho^{\frac{1}{3}} \text{ m/s} \]

For \( \rho \approx 10^{10} \text{ kg/m}^3 \rightarrow 7 \times 10^7 \text{ m/s} \approx 0.25c \)

In effect, star can no longer react to pressure variations at speed \( > C \), so instabilities can set in, neutron degeneracy can be overcome locally, and star collapses to point where \( V_{\text{esc}} > C \)

\[ \Rightarrow \text{ black hole} \]

\[ M_{\text{ch}} \approx 2.2 M_\odot \text{ (non-rotating)} \sim 2.4 M_\odot \text{ (rotating)} \]