Physics 160
Stellar Astrophysics
Prof. Adam Burgasser

Lecture 6
15 October 2013
Lecture 6: Boltzmann & Saha Equations
15 October 2013

PRELIMS
• Announcements [5 min]
• Seminar Summary [5 min]

MATERIAL [70 min]
• [15 min] Line formation (abbrev.)
• [15 min] Statistical distributions
• [15 min] Boltzmann Eqn
• [20 min] Saha Eqn

DEMONSTRATIONS/EXERCISES
• class distribution experiment

MATERIALS
• measuring tapes
Announcements

• HW #3 due Friday @ 5pm in box outside my office SERF 340
  – mistake on #1 – you should be computing n=1, 2 and 3 states; Balmer α = 2->3, Lyman α = 1->2
  – programming bit – random walk experiment

• Local lab => next week (no TA on Friday)

• Astrophysics seminar tomorrow
Dr. Sasha Muratov
The Formation of The First Stars, Galaxies, and Globular Clusters
Line Formation (Review)

- Why wavelengths of high opacity are in absorption
  - Brightness temperature argument
  - Kirchoff’s Laws – treat each wavelength independently

- Energy diagrams for BB, BF, FB, FF absorption & emission
  - QUESTION: Why is it always electrons? – EM force works through charged particles
  - QUESTION: Which of these would be important inside a star?

- Motivation: how do we relate these lines to actual atmosphere properties? What dictates the strength of these lines?

Statistical Distributions

- Number density of particles in Sun’s atmosphere:
  - \( \rho \approx 10^{-3} \text{ kg/m}^3 \) (air = 1 kg/m\(^3\)) [we’ll derive this]
  - \( n \approx \rho/m_H = 10^{24} \text{ m}^{-3} \)

- Number density of photons in Sun’s atmosphere
  - \( n = 2 \times 10^7 \text{ T}^3 \text{ m}^{-3} = 4 \times 10^{18} \text{ m}^{-3} \) [based on Planck Dist.]
  - note \( n_\gamma \ll n \) – low ionization

- Statistical distributions are essential for dealing with such large number of particles/photons

- General form: \( N = \text{int}(f(s)ds) = \text{total number of particles} \) (s = state – position, momentum, spin, energy, spin, etc.

- EXPERIMENT: examine distributions of 1 of:
  - Season of birth
  - Year of birth
  - Height (bring measuring tapes)
  - Neck size (bring measuring tapes)
  - NOTE: small number statistics – is this room representative of UCSD?

- Examples of real distributions
- Maxwell-Boltzmann
  - Motivation for exponential – entropy
  - Derive MB
  - Partition function
- Bose-Einstein/Fermi Dirac -> Planck
- Degenerate gas
- Calculating bulk properties – integrate over distribution

**Boltzmann equation**
- Comes directly from MB probability distribution – but need to include degeneracy
  - NOTE: Ignores fine structure; e.g. spin-flip 21 cm line, Zeeman splitting – this would change degeneracy
    - (add more states)
- Relevance to line absorption – need a certain # of electrons in n=2 state to absorb Balmer Hα
  - Emphasize simplifying steps (e.g., \( x = 13.6 \text{ eV/kT} \))
- Boltzmann equation predicts more electrons in n=2 as T increases (50% point) – Hα strength should rise as we get to hotter T - but doesn’t
- Note: this is pure thermal – collisional excitation can change these ratios

**Saha equation and ionization**
- Motivation: actual strength of lines = [fraction of neutral atoms in state n] x [fraction of neutral atoms to all atoms]
- Derive starting from Boltzmann distribution for free electron states
  - Derive partition function
- Example with H atoms in H I, n=2 state (Balmer Hα) for T = 5000, 10000, 30000 K
  - Emphasize simplifying tasks
- Ripe for computer program!
Main Sequence B8-A2

He I 4026, which is equal in intensity to K in the B8 dwarf β Per, becomes fainter at B9 and disappears at A0. In the B9 star α Peg, He I 4026 = Si II 4129. He I 4471 behaves similarly to He I 4026.

The singly ionized metallic lines are progressively stronger in α CMa and η Oph than in α Lyr. The spectral type is determined from the ratios: B8, B9: He I 4026: Ca II K, He I 4026: Si II 4129, He I 4471: Mg II 4481. A0-A2: Mg II 4481: 4385, Si II 4129: Mn I 4030-4.

from *An Atlas of Stellar Spectra with an Outline of Spectral Classification* by Morgan, Keenan, and Kellman (1943)
Line Formation + Scattering

Spectral classification relies heavily on appearance of spectral lines, which are transitions between bound states, illustrated with an energy diagram.

**Bound-bound**

\[ d = \frac{he}{E_2 - E_1} \]

This is a scattering process, along line of sight we see absorption; in another direction we might see emission.

**Bound-free**

\[ d = \frac{he}{E_1} \]

This is an absorption process, although \( e^- \) can also be captured and emit high-energy photons.

**Collisional Processes**

Photon exchange can also occur between free and bound photons; no emission/absorption, but can populate excited states.
Spectral Line Formation

When we look at individual lines in spectral sequence, we see distinct trends in appearance and disappearance.

Notation:
- Cn II
- ionization
- I = neutral
- II = singly ionized
- III = doubly ionized
- etc.

Hα: transition from n=2 → n=3

For Hα line to form, we need bound e⁻ in the n=2 state - how do we populate this level? By collisional excitation.

To compute line strengths, we therefore need to know:

1. probability & finding free e⁻ with kinetic energy in range that can collisionally excite bound e⁻
2. fraction & atoms with a bound e⁻ in ground state
3. fraction & elements in an ionized state (for Ca II, etc.) - important for H since ionized H has no bound e⁻!
Distributions

As we begin to delve into properties of gas and radiation in stars, we need to appreciate the sheer number of particles involved—this is the regime of statistical physics or thermodynamics.

E.g., Sun's atmosphere: \( \rho \approx 10^{-3} \text{ g/cm}^3 \) (we'll see an easy way to estimate this soon)

Assume all \( \text{H} \) atoms \( \Rightarrow \rho \approx n \cdot m_H = 10^{-3} \text{g/cm}^3 / 1.7 \times 10^{-27} \text{g/cm}^3 \times 10^{24} \text{m}^{-3} \)

Photons: \( n_g \approx 2 \times 10^7 \text{T}^3 \text{m}^{-3} \) (derivable from black distr.)
\[ = 4 \times 10^{15} \text{m}^{-3} \]

To characterize bulk properties of a system, we use distributions of state parameters (anything describing state of system)

\[ n(x) dx = g(x) f(x) dx \]

\( n(x) \) \( \leftrightarrow \) particles
\( g(x) \leftrightarrow \) degeneracy of state \( x \)
\( f(x) \leftrightarrow \) probability of particle being in state \( x \)

Here are some examples:

**Maxwell-Boltzmann**: describes classical (non-relativistic) gas

\[ f(p) \propto e^{-p^2/2kT} \]

or \( f(p) dp \propto e^{-p^2/2kT} \) if \( p^2 dp \)

proportionality constant normalizes \( f \)

\[ \int_0^\infty f(p) dp = N \quad \text{(all particles)} \]

**Bose-Einstein + Fermi-Dirac**: describes quantum system

\[ f(E) dE \propto \frac{1}{e^{(E-x)/kT} + 1} \]

Note for photons, \( \mu_g = 0 \Rightarrow f(E) dE \propto \frac{1}{e^{E/kT} - 1} = \text{Planck function} \]
Degenerate gas when $T \to 0$ (or $kT \ll E_{\text{Fermi}}$)

\[
f(p) \propto \begin{cases} 
1 & p < p_F < \sqrt{2E_F/m} \\
0 & p > p_F
\end{cases}
\]

Fully packed system

Any distribution can be rewritten in terms of other state parameters:

\[
\int g(E)f(E)\,dE = N = \int g(p)f_p(p)\,dp
\]

\[
f_p(E) = \frac{g_p(p)}{g(E)} g(E) f(E) \frac{dp}{dE} = \frac{g_p(p)}{g(E)} \left( \sqrt{\frac{2E}{m}} \right) \sqrt{\frac{2E}{m}} \quad \text{for } E = p^2/2m
\]

Large-scale parameters are then moments of these distributions:

Number of particles

\[
N = \int g(x) f(x)\,dx
\]

Typical momentum

\[
\langle p \rangle = \frac{1}{N} \int g(x) f(x) p(x)\,dx \to \frac{1}{2} \int g(p) f_p(p)\,dp
\]

Typical energy

\[
\langle E \rangle = \frac{1}{N} \int g(x) f(x) E(x)\,dx \to \frac{1}{2} \int g(E) f(E) E\,dE
\]

etc.
Boltzmann Distribution

The most common distribution we will find for systems at high $T$ and low $p$ (we'll see the limits of this later) is that of the Boltzmann Distribution

$$\frac{P(s_2)}{P(s_1)} = e^{-\frac{(E_1 - E_2)}{kT}}$$

probability of system/particle occupying state $2$ vs. energy of state

This emerges from recognizing that the number of available states is a measure of entropy:

$$S \propto \ln \Omega$$

$\Omega$ = number of states

Consider a system that is immersed in a large reservoir that maintains the temperature and supplies available states. Then the probability that our system is in state $S_1$ is just proportional to the number of available states.

$$P(s_1) \propto \Omega_1(s_1)$$

$$\frac{P(s_1)}{P(s_2)} = e^{\frac{E_1 - E_2}{kT}}$$

entropy of reservoir for states $1 \& 2$

1st Law of Thermodynamics:

$$dS = \frac{1}{T} (dU + pdV - \mu dN)$$

inertial energy

expansion/contraction

$$dS = \frac{U}{T} = -\frac{E_{\text{state}}}{T}$$

(constant total energy: $U + \mu N$ is constant)

$$\frac{P(s_1)}{P(s_2)} = e^{\frac{-(E_1 - E_2)}{kT}}$$

We have to account for the fact that in each state, there may be multiple "sites" available to per particle - this is degeneracy $g_i$. So $N_i = P(s_i)g_i$

$$\frac{N_i}{N_j} = \frac{g_i}{g_j} e^{\frac{-(E_1 - E_2)}{kT}}$$

this is the Boltzmann Eqn
The probability of any one state can also be written as
\[ P(s) = \frac{1}{Z} e^{-E_s/kT} \]

where \[ \int P(s) ds = \Sigma P(s) = \frac{1}{Z} \Sigma e^{-E_s/kT} = 1 \]

\[ \Rightarrow Z = \Sigma e^{-E_s/kT} \] is the partition function or sum over all states folding in our degeneracy:

\[ N(s) = \frac{1}{Z} \sum e^{-E_s/kT} \]

\[ \Rightarrow Z = \sum \frac{1}{N_{TOT}} \sum e^{-E_s/kT} \]

Why is this useful? Consider neutral H atoms in the Sun's atmosphere. Bound e\(^-\) can occupy orbitals based on their excitation energy:

\[ E_n = -13.6 \text{ eV/n}^2 \]

Degeneracy for these states is based on orbital angular momentum \( j \) and spin \( S = \frac{1}{2} \) or \( j = \frac{1}{2}, \frac{3}{2}, ... \)

\[ j_n = \frac{1}{2} \left( n - \frac{1}{2}, n + \frac{1}{2}, ... \right) \]

Therefore:

\[ \frac{N_i}{N_j} = \left( \frac{i}{j} \right)^2 \cdot \frac{13.6 \text{ eV/} j^2}{\text{kT}} \]

\[ N_i \approx \frac{2}{\pi} \cdot \frac{13.6 \text{ eV/} i^2}{\text{kT}} \]

\[ Z = \frac{2}{N_{TOT}} \cdot \frac{\Sigma e^{13.6 \text{ eV/} j^2 \text{kT}}}{N_{TOT}} \]

For Sun, \( kT \approx (8.6 \times 10^{-5} \text{ eV/}k)(5800 \text{ K}) \approx 0.5 \text{ eV} \)

Consider \( Z \):

\[ Z = \frac{2}{N_{TOT}} \left( e^{27} + 9 e^{60} + 9 e^{300} + ... \right) \]

\[ \approx \frac{10^{12}}{N_{TOT}} \left( 1 + 7 \times 10^{-9} + 3 \times 10^{-10} + ... \right) \]

\[ \Rightarrow N_i \approx N_{TOT} \left( 2 \times 10^{-27} \right) e^{27} \approx N_{TOT} \]

\[ N_2 \approx N_{TOT} \left( 2 \times 10^{-27} \right) 4 e^{60} \approx 7 \times 10^{-9} N_{TOT} \text{ etc...} \]

\[ \Rightarrow \frac{N_o}{N_1} \approx 4 e^{27} \left( 1/4 - 1 \right) \approx 6 \times 10^{-9} \approx \text{roughly the same} \]
Saha Equation

Our expectation from Boltzmann Eqn is that higher order transitions (e.g. 3 → 2, 4 → 3) should be increasingly more common in hotter stars due to greater probability of e⁻ recombination at those levels.

We need to calculate ratio of ionized atoms to neutral atoms — try

Boltzmann:

\[ \frac{d n^+}{n} \rightarrow \frac{d y^+}{g_1} \approx \frac{g_e}{g_1} e^{-\frac{(E^+-E_1)}{kT}} \]

\[ E^+ = \frac{p^2}{2m} \Rightarrow E^+ = E_1 + \frac{p^2}{2m} \]

\[ \Rightarrow \text{transition} \]

\[ \text{potential} \]

What is \( d n^+ \)? ions that are free and have \( p \in [p, p+dp] \) and ground state H lines

What is \( d g^+ \)? \( g^+ = g_{H^+} + g_{e^−} \) unbound e⁻ states

For free state, consider counting e⁻ in 6D phase space box \( d^3p \cdot d^3x \)

Uncertainty principle sets a minimum box size:

\[ \Delta^3 \times \Delta^3 p = h^3 \]

\[ \Rightarrow \text{how many if there is} \]

\[ \text{6D volume?} \]

\[ \# \text{states} = \frac{d^3 \cdot d^3 p}{h^3} \]

\[ \Rightarrow \int \text{over spatial volume} \]

\[ \Rightarrow \text{volume} \]

\[ \Rightarrow \text{number} \]
\[ \frac{dn^+}{n_1} = \frac{g_i^+}{g_i} \left( \frac{8\pi}{n e h^3} \right) e^{-\left( \frac{E_i + \frac{1}{2} m_e v^2}{kT} \right)} \]

Integrate over both sides:

\[ \frac{n^+}{n_1} = \frac{g_i^+}{g_i} \left( \frac{8\pi}{n e h^3} \right) e^{-\frac{\chi u}{kT}} \int_0^\infty e^{-\frac{p^2}{2m_e kT}} p^2 dp \]

Substitute \( x = \frac{p^2}{2m_e kT} \), \( p^2 = 2m_e kT x \), \( dp = \sqrt{2m_e kT} dx \)

and \( \int_0^\infty e^{-x^2} x^2 dx = \frac{\sqrt{\pi}}{4} \)

\[ \Rightarrow \frac{n^+}{n_1} = \frac{2g_i^+}{g_i} \left( \frac{2\pi m_e kT}{h^2} \right)^{\frac{3}{2}} e^{-\frac{\chi u}{kT}} \]

This is the number of ionized atoms to ground state neutral atoms - we want to extrapolate to all states - use partition function!

\[ n_1^+ = \frac{1}{2} g_i^+ e^{-\frac{E_i}{kT}} \]
\[ n_1^- = \frac{1}{2} g_i^- e^{-\frac{E_i}{kT}} \]
\[ n_1 = \frac{1}{2} g_i e^{-\frac{E_i}{kT}} \]
\[ \tilde{Z} = \sum_i \frac{1}{2} g_i e^{-\frac{E_i}{kT}} \]

Note that \( n^0 = \sum n_i^0 = n_1^0 e^{\frac{\tilde{Z}}{g_i}} \Rightarrow n_i^0 = \frac{\tilde{Z}}{g_i} e^{-\left( \frac{E_i + \frac{1}{2} m_e v^2}{kT} \right)} \cdot e^{-\frac{\chi u}{kT}} \)

Saha Equation

For sequential ionization states:

\[ \frac{n_i}{n_i^-} = \frac{Z_i^+}{Z_i^-} \left( \frac{2\pi m_e kT}{h^2} \right)^{\frac{3}{2}} e^{-\frac{\chi u}{kT}} \]

First derived in 1920 by Meghnad Saha, Indian astronomer. Used by Cecilia Payne-Gaposchkin to determine that stars were made of hydrogen (1925 Thesis).
Application to H²⁺

Problem: Compute fraction of all H atoms (H⁺, H²⁺) that has H⁺ up e⁻ in n=2 state. This fraction is proportional to strength of Balmer line.

\[ \frac{N_{H⁺}(n=2)}{N_H} = \frac{N_{H⁺}(n=2)}{N_H} \]

Excitation correction

Step 1: Compute partition function

\[ \tau = \sum_{j=0}^\infty g_j e^{-(E_j-E_0)/kT} \]

\[ \tau = 1 \quad \text{no bound e⁻ in H⁺, so no degeneracy!} \]

\[ \tau_0 = g_1 + g_2 e^{-(E_2-E_1)/kT} + \cdots \]

\[ E_2 - E_1 = -3.4 + 13.6 = 10.2 \text{ eV} \]

\[ kT = 0.66T_4 = 0.3 \text{ M}_\text{star} \]

\[ 3:0 \quad 0 \text{ star} \]

\[ e^{-\frac{(E_2-E_1)/kT}{\tau}} = 2 \times 10^{-15} \text{ M}_\text{star} \]

\[ = 0.05 \text{ O star} \]

Can generally ignore higher order terms

\[ \tau_0 = 2 + 2(2)^2 [\text{small e⁻}] \approx 2 \]

\[ \frac{n(\text{H}⁺)}{n(\text{H}⁺)} = \frac{2}{n_e(2)} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\frac{13.6}{0.66T_4}} = \frac{2.4 \times 10^{21}}{n_e} T_4^3 \]

for \( T = 10,000 \text{ K} \quad n_e = 10^{14} \text{ cm}^{-3} \quad \frac{n(\text{H}⁺)}{n(\text{H}⁺)} \approx 1 \]

\[ \Rightarrow \chi_{\text{lum}} \approx \frac{n(\text{H}⁺)}{n(\text{H}⁺)+n(\text{H}⁺)} \approx \frac{1}{2} \quad \text{50% of H atoms are ionized} \]

in a C² atmosphere

Note: \( kT \approx 0.9 \text{ eV} < 13.6 \text{ eV} \)

Only a tiny fraction of e⁻ have KE high enough for ionization. Larger \( Z \) reflect less e⁻ & \( e⁻ \) not available to the e⁻.
Example: H I + Ca II in Sun

Hα (H I ν=3-2→2) and Ca II Hβ line (Ca II ν=1-0→2) are key features
in Solar photosphere. How do the strengths of these lines relate to H+Ca
abundances?

Solar photosphere: \( T \approx 5800 \text{ K} \)
\[ n_e \approx 10^{13} \text{ cm}^{-3} \]

\[ \frac{n(H^+)}{n(H^0)} = \frac{28 \times 10^{15}}{10^{13}} (5800)^{3/4} e^{-1.6 \times 10^5/T} \approx 8 \times 10^{-9} \Rightarrow \text{most H is neutral} \]

\[ \frac{n_2}{n_H} = \frac{n_{H^+}}{n_{H^0}} \approx \frac{n_{e^+}}{n\text{H}^0} \]

\[ \left( \frac{n_2}{n_1} \right)^2 = 1 + \frac{n_2}{n_1} \times \frac{n_2}{n_1} = \frac{3}{2} e^{-\frac{1}{2} (13.6 - 3.4)/0.66 T_H} = 5 \times 10^{-9} \Rightarrow \text{drop higher terms} \]

\[ \frac{n_2}{n_1} \approx 5 \times 10^{-9} \Rightarrow \text{very few e}^- \text{in} \ n_2 \text{ state} \]

\[ \Rightarrow \frac{n_2}{n_H} \approx \frac{n_{e^+}}{n_{H^0}} \approx 5 \times 10^{-9} \Rightarrow \text{Hα is relatively weak} \]

Ca II

note: \( 2 \nu_1 = 1.32 \quad 7 \text{ Ca} \quad 2.3 \)

\[ \chi_{\text{ion}} = 6.11 \text{ eV} \]

\[ \frac{n(Ca II)}{n(Ca I)} = 2.3 \times 10^{15} (5800)^{3/4} e^{-6.11/0.66 \times 1.58} \]

\[ \approx 1800 \Rightarrow \text{most Ca I is ionized!} \]

\[ n_1 \rightarrow 2 \quad \Delta E = 3.82 \text{ eV} \quad g_1 = 2 \quad g_2 = 4 \quad (\nu_2 \text{ Ca} \text{II}) \]

\[ \Rightarrow \left( \frac{n_2}{n_1} \right)^2 = e^{-\frac{1}{2} (3.12/0.66 \times 1.58)} = 3.8 \times 10^{-3} \]

\[ \Rightarrow \text{most Ca II in ground state} \]

\[ \frac{H\alpha}{Ca \text{II}} = \frac{n_{H^+} n_2}{n_{Ca^+} n_1} \]

\[ \frac{n_2}{n_{H^+}} = n_{H^0} \times \frac{n_2}{n_{H\text{II}}} \times \frac{n_{e^+}}{n_{H\text{II}}} \times \frac{n_{e^+}}{n_{H^0}} \times \frac{n_{e^+}}{n_{H\text{II}}} \times \frac{n_{e^+}}{n_{H\text{II}}} \times \frac{n_{e^+}}{n_{H^0}} \times \frac{n_{e^+}}{n_{H\text{II}}} \times \frac{n_{e^+}}{n_{H^0}} \]

\[ \Rightarrow \text{H is very more abundant} \]

(then \( n_{Ca} \) & the element)
Spectral Line Profiles

If a band-band transition can only occur at a specific energy/wavelength, one would have infinitely sharp lines. In reality, lines are broadened by a variety of mechanisms.

1. **Natural line broadening**— quantum mechanics implies an inherent uncertainty in energy of photon and its lifetime through Planck constant:

   \[ \Delta \omega \Delta t = \hbar \]

   \[ \Rightarrow \text{energy levels are "fuzzy" and have a finite lifetime for making transition} \]

   \[ E = \frac{hc}{\lambda} \Rightarrow |\Delta E| = \frac{hc}{\lambda^2} \Delta \lambda = \frac{2\pi c}{\lambda^2} \Delta \omega \Delta t \]

   \[ \Delta \omega = \frac{2\pi c}{\lambda^2} \frac{1}{\Delta t_{\text{trans}} + \Delta t_{\text{inst}}} \]

   \[ \Rightarrow \Delta \omega \approx 9 \times 10^{-12} \text{ cm}^{-1} \]

2. **Doppler broadening**— we observe atoms in motion in a gas, so observed \( \Delta \omega \) is shifted by Doppler effect:

   \[ \Delta \omega = \frac{\Delta v}{c} = \frac{\Delta v \cdot \Delta t_{\text{trans}}}{c} \]

   \[ \text{averaging over a uniform gas with } \bar{V}_n = \sqrt{\frac{2kT}{m}} \text{ (Maxwell-Boltzmann dist)} \]

   \[ \Rightarrow \Delta \omega = \frac{2\pi c}{\lambda^2} \sqrt{\frac{2kT \ln 2}{m}} \]

   \[ \Rightarrow \Delta \omega_{\text{d}} = \frac{2\pi c}{\lambda^2} \sqrt{\frac{2kT \ln 2}{m}} \]

   \[ \text{directional term} \]

   \[ \Rightarrow \Delta \omega_{\text{d}} = \frac{2\pi c}{\lambda^2} \sqrt{\frac{2kT \ln 2}{m}} \]

   \[ \text{there may also be large-scale motion caused by turbulence, convection, etc.} \]

   \[ \Rightarrow \Delta \omega_{\text{d}} = \frac{2\pi c}{\lambda^2} \sqrt{\frac{2kT \ln 2}{m} + \bar{V}_0^2} \]

   \[ \text{important for cool stars with photospheric convection (giants, supergiants, etc.)} \]
Pressure Broadening - "Collisions" or "Near collisions" of atoms cause ion electric fields to interact, changing atomic orbitals. This can shift line transitions.

\[
\Delta \lambda \approx \frac{d^2}{v_{\text{thermal}} \sqrt{m}} \Rightarrow \Delta \lambda \approx \frac{d^2}{c} \frac{\nu_0}{T} \sqrt{\frac{2 \nu T}{m}}
\]

Note: similar in form to natural line broadening, and in many stars of same magnitude, not true for stars with dense photospheres (e.g. brown dwarfs, white dwarfs)

Line profile forms

Natural + pressure broadening leads to Lorentz profiles:

\[
f_L(d; d_0, \Delta \lambda) \propto \frac{1}{\pi} \left[ \frac{\Delta \lambda c}{(d-d_0)^2 + \Delta \lambda_c^2} \right] \frac{d\lambda}{d\nu}
\]

Doppler broadening produces an exponential profile:

\[
f_D(d) \propto \frac{\sqrt{m c^2}}{2 \hbar k T d_0^2} \exp\left(-\frac{m c^2 (d-d_0)^2}{2 \hbar^2 k T d_0^2}\right) \frac{d\nu}{d\lambda}
\]

The combined profile integrates these into a Voigt profile:

\[
f_V(d) \propto \int_{-\infty}^{\infty} f_L(d-d', \Delta \lambda) f_D(d', T) \frac{d\lambda'}{d\nu} \, d\nu'
\]

must consider combined effects for every possible transition.