Physics 160
Stellar Astrophysics
Prof. Adam Burgasser

Lecture 9
Stellar Energy Sources
22 October 2013
Announcements

• HW #4 due tomorrow @ 5pm in box outside my office SERF 340

• Schedule shuffle: Energy transfer next Tuesday, separate lecture on Brown Dwarfs: Monday 10/28 3-4pm @ Mayer 2623

• no HW next week, but energy transfer will be on...

• **Midterm: Thursday 10/31 in class**
  – equation sheet now online

• Local lab: postpone to Monday 10/28 7-9pm at Torrey Pines Glider Port *WEATHER PERMITTING* (email confirmation will be sent)

• Mid-quarter evaluations

• Looking ahead: project proposals due 7th week – information sheet now on course webpage
Brown dwarfs got booted!
to Monday 10/28 3-4pm Mayer 2623

Revised Schedule

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<td>The Sun: interior structure, photosphere, magnetosphere</td>
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Review
• 3 stellar structure equations
• Chemical vs. Gravitational vs. Nuclear – compare energy densities = energy/mass
  o Chemical: $10^8$ J/kg
  o Gravitational: $10^{11}$ J/kg (to Sun’s size... )
  o H->He fusion: 27 MeV/4u = $6 \times 10^{14}$ J/kg
  o Cf. fission
    ▪ $^{238}$U = 4.7 MeV/238 u -> $2 \times 10^{12}$ J/kg
    ▪ $^{40}$K = 1.5 MeV/40 u -> $3 \times 10^{12}$ J/kg
  o Best is pure mass: $E/m = c^2 = 10^{17}$ J/kg (matter-antimatter rxns)
• pp chain
  o neutrino loss
  o some important links in the chain
  o CNO chain – same process, different actors

Nuclear reaction rates
• Luminosity -> energy generation -> energy equation for stellar structure
• rxn rate -> Cross section approach
• $<\sigma v>$ from MB distribution
• Coulomb barrier revisited – quantum barrier
• Gamow peak (tail of distribution)
• Common parameterization
• Reaction rate equations?
  o general form and dependence on abundances
  o reaction chain

Polytrope model
• Can use 2 equations of stellar structure with a parameterized polytrope EOS to solve
• Derivation
Lecture 9: Stellar Interiors: Energy Generation and Transfer, Polytrope Models
24 October 2013

PRELIMS
• Announcements [10 min]

MATERIAL [70 min]
• [15 min] Review
• [30 min] Energy generation rate
• [15 min] Polytrope models

DEMONSTRATIONS/EXERCISES
• none

MATERIALS
• none
• Boundary conditions
• Solutions and zeropoints (analytic)
• Relation to mass, radii, core pressure, average to core density
CNO cycle for H fusion

\[ ^{12}\text{C} + p \rightarrow ^{13}\text{N} + \gamma \]
\[ ^{13}\text{N} \rightarrow ^{13}\text{C} + e^+ + \nu_e \]
\[ ^{13}\text{C} + p \rightarrow ^{14}\text{N} + \gamma \]
\[ ^{14}\text{N} + p \rightarrow ^{15}\text{O} + \gamma \]
\[ ^{15}\text{O} \rightarrow ^{15}\text{N} + e^+ + \nu_e \]
\[ ^{15}\text{N} + p \rightarrow ^{12}\text{C} + ^4\text{He} \]

http://www.astronomy.ohio-state.edu/~pogge/Ast162/Unit2/energy.html
CNO cycle for H fusion

\[ {^{12}\text{C}} + p \rightarrow {^{13}\text{N}} + \gamma \]
\[ {^{13}\text{N}} \rightarrow {^{13}\text{C}} + e^+ + \nu_e \]
\[ {^{13}\text{C}} + p \rightarrow {^{14}\text{N}} + \gamma \]
\[ {^{14}\text{N}} + p \rightarrow {^{15}\text{O}} + \gamma \]
\[ {^{15}\text{O}} \rightarrow {^{15}\text{N}} + e^+ + \nu_e \]
\[ {^{15}\text{N}} + p \rightarrow {^{12}\text{C}} + ^4\text{He} \]
Main sequence

stellar death

B²FH (1957)
Nuclear reaction rates

Now that we know nuclear reactions happen, it's important to assess how rapidly they occur and hence how much they contribute to overall energy budget. Remember, to avoid further contraction & a star on the main sequence, we must have

\[ L = L_{\text{nucl}} = \sum L_{\text{nucl, individual reactions}} \]

We can parameterize the last quantity in terms of mass.

\[ \frac{dL_i}{dt} = \varepsilon_i dm \]

\[ \text{energy production rate for reaction } i \]

unit: energy/time/mass

(in principle, can also include gravity)

\[ dm = \rho(r) 4\pi r^2 dr \]

\[ \Rightarrow \quad \frac{dL_i}{dr} = 4\pi r^2 \varepsilon_i(r) \rho(r) \quad \text{or} \quad \frac{dL_i}{dr} = 4\pi r^2 \varepsilon_i(r) \rho(r) \quad \text{where} \quad \varepsilon_i(r) = \sum \varepsilon_i(\text{all terms}) + \text{terms due to gravity} \]

Each reaction produces an energy \( E_0 \)

\[ \Rightarrow \quad \varepsilon_{ij} = \left( \frac{E_{0,ij}}{\rho N} \right) \sigma_{ij} \quad \text{reaction rate between nuclei } i + j \]

with: reactions/time/volume

Consider classic "reaction": two particle striking each other

\[ \sigma \quad \text{cross section (area \( \sigma \) units distance}^2 \)\]

\[ \text{every incident particle } i \text{ that approaches within area } \sigma \text{ strikes particle } j \]

This can be interpreted as a probability of interaction (within \( \sigma \) probability \( = 1 \))
Now consider a set of incident particles with energy \( E \in [E_0, E_0 + dE] \).

\[ ds = V(E) \, dt \]

\[ \frac{\text{reaction}}{\text{time} \times \text{volume}} = \frac{\text{reaction}}{\text{time} \times \text{volume}} \times \frac{\text{target nuclei}}{\text{volume}} = \frac{n_i n_j}{n} \sigma(E) V(E) \eta \frac{dE}{E} \]

\[ N_i \eta \frac{dE}{E} = \frac{n_i n_j}{n} \sigma(E) V(E) \eta \frac{dE}{E} \]

\[ n_i \eta = \frac{n_i n_j}{n} \sigma(E) V(E) \eta \]

This is per target nucleus; hence, for all target nuclei:

\[ \frac{\text{reaction}}{\text{time} \times \text{volume}} = \frac{\text{reaction}}{\text{time} \times \text{volume}} \times \frac{\text{target nuclei}}{\text{volume}} = \frac{n_i n_j}{n} \sigma(E) V(E) \eta \frac{dE}{E} \]

Note that \( n = \int_0^\infty n(E) \, dE = \int_0^\infty n(u) \, du \).

Recall \( n(u) \, du = n \left( \frac{M}{2\pi kT} \right)^{3/2} e^{-M/u^2} \frac{8\pi}{M} u^2 \, du \).

Letting \( E = \frac{1}{2} M v^2 \)

\[ n(E) = n \left( \frac{M}{2\pi kT} \right)^{3/2} e^{-E/UT} \frac{8\pi}{M} E \]

But \( n(E) \, dE = n(u) \, du \).

\[ n(E) = n(u) \frac{dE}{du} (\frac{3}{2} \frac{d(\sqrt{u})}{du}) \frac{1}{\sqrt{u}} \frac{1}{\sqrt{4\pi M E}} \]

\[ \Rightarrow n(E) \, dE = n \frac{2}{\sqrt{\pi}} (kT)^{3/2} e^{-E/UT} E \, dE \quad \text{and} \quad V(E) = \sqrt{\frac{\pi M}{4\pi M E}} \]

So \( n_i j = n_i n_j \left( \frac{2}{\sqrt{\pi}} \right)^{3/2} \frac{1}{\sqrt{\pi M}} \int_0^E \sigma(E) e^{-E/UT} E \, dE \)

What is \( \sigma(E) \)?

1. "Size" of nucleus is deBroglie \( \lambda = \frac{h}{p} \)

2. \( \sigma(E) \propto \pi \lambda^2 \propto \left( \frac{h}{m} \right)^2 \propto \frac{1}{E} \)

3. Probability of reaction \( \Rightarrow \) quantum tunneling.


Classically, nuclei see a repulsive potential due to Coulomb barrier

\[ U(r) = \frac{Z_1 Z_2 e^2}{r} \approx \frac{Z_1 Z_2}{R_{fm}} \text{ MeV} \]

However, within region of strong nuclear force \( < 1 \mu \approx 10^{-15} \text{ m} \), strong attraction potential

Classically, \( KE > U_c \) to penetrate

\[ \frac{1}{2} \mu v^2 \approx \frac{3}{2} kT > \frac{Z_1 Z_2 e^2}{r} \]

\[ \Rightarrow T > \frac{2}{3} \frac{Z_1 Z_2 e^2}{k} \]

For \( Z_1 = Z_2 = 1 \), \( T > 10^{10} \text{ K} \)

Alternative: quantum tunneling

DeBroglie wavelength \( \lambda = \frac{\hbar}{p} \)

\[ \langle KE \rangle = \frac{p^2}{2 \mu} = \frac{(\hbar/\lambda)^2}{2 \mu} \approx \frac{Z_1 Z_2 e^2}{r} \approx \frac{Z_1 Z_2 e^2}{\lambda} \]

\( \Lambda \) assumes to get within DeBroglie \( \lambda \)

\[ \Rightarrow \lambda = \frac{h^2}{2 \mu Z_1 Z_2 e^2} \text{ critical length scale} \]

\[ \Rightarrow T > \frac{2}{3} \frac{Z_1 Z_2 e^2}{\hbar^2} = \frac{4}{3} \frac{\mu (Z_1 Z_2 e^2)^{\frac{1}{2}}}{\hbar^2} \]

(For \( Z_1 = Z_2 = 1 \) and \( \mu = \frac{m_p}{2} \))

\[ \Rightarrow T \approx 10^7 \text{ K} \]

\( \uparrow \) about right

for Sun
Finite barrier tunneling problem

1D Schrödinger Eqn

\[-\frac{\hbar^2}{2\mu} \frac{d^2\Psi}{dx^2} + U(x)\Psi = E\Psi\]

\[\Rightarrow \frac{d^2\Psi}{dx^2} = -\frac{2\mu}{\hbar^2} (E - U(x))\Psi\]

General sol'n \(\Psi(x) = A e^{ikx} + B e^{-ikx}\)

\[k^2 = \frac{2\mu}{\hbar^2} (E - U(x))\]

Two cases:

- \(k > 0 \Rightarrow E > U(x)\) free particle
- \(k < 0 \Rightarrow E < U(x)\) exponential decay \(\Psi(x) = A e^{-kx}\)

\[\Rightarrow\] probability & propagation \(\propto e^{-k\Delta r}\)

\[-\sqrt{\frac{2\mu}{\hbar^2} (U(x) - E)} \Delta r\]

\[\Rightarrow e^{-\frac{\sqrt{2\mu (U(x) - E)} \Delta r}{\hbar^2}}\]

\[-2\pi^2\rho m / \hbar \text{ de Broglie length } = h / p\]

Recall \(\frac{1}{2} \mu v^2 = \frac{p^2}{2\mu} = \frac{7_1^2 e^2}{r_{\text{min}}} = \text{ minimum classical energy}\)

\[r_{\text{min}} = \frac{2\mu Z_1 Z_2 e^2}{p^2}\]

\[\rho = r_{\text{min}} / \hbar = \frac{2\mu Z_1 Z_2 e^2}{hp}\]

\[\propto \frac{Z_1 Z_2 e^2}{h^2} \propto E^{-1/2}\]

\[\] probability \(\propto e^{-bE^{1/2}}\) where \(b = \frac{2\pi \sqrt{2\mu Z_1 Z_2 e^2}}{h}\)

\[\approx 31.3 \times Z_1 Z_2 \sqrt{A} (\text{meV})^{1/2}\]

\[\omega / A = \frac{A_1 A_2}{A_1 + A_2} (\text{at mass})\]

Of course, probability of penetration higher with higher \(E\).
back to reaction rate:

\[ \sigma(E) \propto \frac{1}{E} \quad \text{and} \quad \sigma(E) \propto E^{-bE^{\frac{1}{2}}} \]

\[ \Rightarrow \sigma(E) \propto \frac{S(E)}{E} e^{-bE^{\frac{1}{2}}} \]

\[ r_{ij} = n_i n_j \left( \frac{2}{\hbar} \right)^{2} \frac{1}{\sqrt{\mu \tau}} \int_{0}^{\infty} S(E) E^{-bE^{\frac{1}{2}} - \frac{E}{\hbar \tau}} \, dE \]

What does this combination look like?

\[ \text{peak energy: } \frac{d}{dE} \left( e^{-bE^{\frac{1}{2}} + E/\hbar \tau} \right) = 0 \quad \Rightarrow \quad E_0 = \left( \frac{b \hbar \tau}{2} \right)^{2/3} \]

\[ E_0 \approx \frac{(\pi^2 \sqrt{2} n \hbar^2 e^2 \hbar \tau)^{2/3}}{n} \]

\[ E_0 = 1.22 \left( \frac{2^2 \pi^2 e^2 \hbar \tau}{n} \right)^{2/3} \text{ keV} \]

\[ A = \frac{A_1 A_2}{A_1 + A_2} \quad T_6 = \frac{T}{10^6} \text{ K} \]

Parameterization: commonly write \( r_{ij} \propto e_0 X_i X_j \rho^\alpha T^\beta \Rightarrow E_i j \propto e_0 X_i X_j \rho^\alpha T^\beta \)

E.g., PP chain: \( E_{pp} \approx e_{0,pp} \rho X^2 T_6^{4/3} \quad e_{0,pp} = 1.96 \times 10^{-11} \text{ W m}^{-2} \text{ kg}^{-1} \)
Fig. III,1. Energy generation in ergs g\(^{-1}\) sec\(^{-1}\) as a function of temperature in the \(pp\) chain and the CN cycle. The ordinates give loge directly for \(\rho x_H^2 f_{pp} = 100 \text{ g/cm}^3\), \(\rho x_{H^2} x_N f_N = 1 \text{ g/cm}^3\) or \(\rho x_{H^2} x_O f_O = 1 \text{ g/cm}^3\). They are thus appropriate for \(x_N = x(N^{14})\) or \(x_C = x(C^{12}) \approx 0.8(x(C^{12} + C^{13}) = 0.01 x_H\). If \(N^{14}(p, \gamma)\) is resonant the CN-cycle rate is determined by \(C^{12}(p, \gamma)\) and \(x_C \approx 0.8 x_{CN}\). If \(N^{14}(p, \gamma)\) is nonresonant, it determines the CN-cycle rate and \(x_N \approx x_{CN}\).
Nuclear reaction chains

Nucleosynthesis + decay can proceed along several chains through step, mitigated by EM, strong + weak forces

![Nuclear Reaction Diagram]

1. p capture (strong)
2. n capture (strong)
3. e⁻ capture (weak)
4. e⁺ capture (strong)
5. p decay (weak)
6. α capture (weak)
7. α decay (strong)

Weak interactions are as they sound - weak, & unlikely, so these tend to be rate-limiting steps. Also e⁻/e⁺ emission is accompanied by E/γ emission, and neutrino energy is generally lost from star (except in ultradense cores undergoing SN shock).

Most reactions are hybrid, in that reactants → products through and intermediate channel(s):

\[ ^{1}\text{H} + ^{1}\text{H} \rightarrow ^{2}\text{He}^{*} \rightarrow ^{1}\text{H} + ^{0}\text{e} + \nu_{e} \]

proton capture beta decay (strong) (weak)

↑ rate limiting step for total ran
Reactions + Abundances

Using these various chains, it is possible to infer the time-dependent abundances of elements at various regions in a star, keeping in mind that convective mixing may even these abundances out.

Let's consider the PPI chain alone:

\[
\begin{align*}
1 \text{H} + 1 \text{H} & \rightarrow 2 \text{H} + \text{e} + \nu_e \\
2 \text{H} + 2 \text{H} & \rightarrow 3 \text{He} + \text{e} \\
3 \frac{1}{2} \text{He} + 3 \frac{1}{2} \text{He} & \rightarrow 4 \frac{1}{2} \text{He} + 2 1 \text{H}
\end{align*}
\]

Here \([x]\) is the number density of species \(x\) and \(\Delta x\) is just \(<\sigma v> dx\), a fraction primarily of temperature.

If this is a closed cycle then the number densities vary by:

1. \[
\frac{d[\text{H}]}{dt} = -2 \text{app} \frac{[\text{H}]}{2} - \text{pd} \text{[H]}[\text{D}] + 2 \frac{1}{3} \frac{[\text{He}]}{2}
\]
2. \[
\frac{d[\text{He}]}{dt} = \text{app} \frac{[\text{H}]}{2} - \text{pd} \text{[H]}[\text{D}]
\]
3. \[
\frac{d[\frac{1}{2} \text{He}]}{dt} = \text{pd} \text{[H]}[\text{D}] - 2 \frac{1}{3} \frac{[\text{He}]}{2}
\]
4. \[
\frac{d[\frac{3}{2} \text{He}]}{dt} = \frac{[\frac{3}{2} \text{He}]}{2}
\]

These are four simultaneous equations & four unknowns (assuming \(d\) is calculated), and assuming one starts with initial abundances, you can in principle determine abundances over time.
In addition, for "carrier" elements, one can determine steady state abundances; take [D] for instance:

\[
\frac{d[D]}{dt} = 0 = A_{pp} \left( \frac{[H]^2}{2} \right) - A_{pd} ([D][H])
\]

\[
\Rightarrow \frac{[D]}{[H]} = \frac{A_{pp}}{2A_{pd}}
\]

Recall that by approximating the non-resonant reaction rates as Gaussians over the Gaussian peaks, we get:

\[
d = \langle \sigma v \rangle = \frac{7.2 \times 10^{-19}}{A_{pd} A_{pp}} \frac{S_{pp}}{S_{pd}} \left( \frac{A_{pp}}{A_{pd}} \right)^{2/3} e^{-\frac{\tau}{T_0}} \text{ cm}^3 \text{s}^{-1}
\]

\[
\Rightarrow \tau = \frac{350 \text{s}}{T_0}
\]

\[
= \frac{[D]}{[H]} = \frac{1}{2} \frac{S_{pp}}{S_{pd}} \frac{A_{pp}}{A_{pd}} \left( \frac{A_{pp}}{A_{pd}} \right)^{2/3} \left( \frac{7^{1/2} A}{T_0} \right)^{\frac{1}{2}}
\]

\[
< 42.4 \times T_0^{-1/3} [A_{pp} - A_{pd}]
\]

For Sun, \( T_0 = 15 \), \( A_{pp} = 1/2 \), \( A_{pd} = 2/3 \), \( S_{pp} = 3.8 \times 10^{-22} \text{cm}^3\text{cm}^{-3} \text{ s}^{-1} \)

\[
S_{pd} = 2.5 \times 10^{-4} \text{cm}^3\text{cm}^{-3} \text{ s}^{-1}
\]

\[
\Rightarrow \frac{[D]}{[H]} \approx 3 \times 10^{-18}
\]

More complex chains of reactions can be computed; see the famous

\( \text{C} + \text{H} \rightarrow \text{C}_2 \text{H} \rightarrow \text{B}-\text{Bridges, B}-\text{Bridges, Fowler + Hoyle (1957)} \)

Note that if \([D] = [^{3}\text{He}]\) are in equilibrium, then

\[
\frac{d[^{4}\text{He}]}{dt} = \frac{1}{4} A_{pp} [H]^2
\]

i.e., production rate depends solely on rate to fuse \( \text{H} \) atoms
Polytrope models

Considerable simplicity can be gained if one assumes a functional form for the equation of state called a polytrope:

\[ p = K \rho^\gamma \]

Start with hydrostatic eqn. Eqn:

\[ \frac{dP}{dr} = \frac{-GM(r)p}{r^2} \Rightarrow \frac{r^2 dP}{\rho dr} = -GM(r) \]

\[ \Rightarrow \frac{d}{dr} \left( \frac{r^2 dP}{\rho dr} \right) = -G \frac{d\rho}{dr} \]

\[ \Rightarrow \frac{d}{dr} \left( \frac{r^2 dP}{\rho dr} \right) = -4\pi G \rho \]

\[ \Rightarrow \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dP}{\rho dr} \right) = -4\pi G \rho \]

This is actually \[ \nabla^2 \Phi = 4\pi G \rho \] Poisson’s equation

where \[ \Phi = -\frac{GM(r)}{r} \] specific gravitational potential

Cf. \[ \nabla^2 \Phi = -4\pi \rho \] in electrodynamics, where \[ \vec{E} = -\nabla \Phi \]

This form applies for any \( 1/r \) potential field

Now assume \( p = K \rho^\gamma \)

\[ \Rightarrow \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 dP}{\rho dr} \right) = K \gamma \frac{d}{dr} \left( \frac{r^2 \gamma p^{\gamma-2} dP}{\rho dr} \right) = -4\pi G \rho \]

Let \[ \gamma = (n+1)/n \] where \( n \) is polytropic index

\[ \Rightarrow \frac{n+1}{n} \frac{K}{\rho} \frac{d}{dr} \left( \frac{r^2 dP}{\rho dr} \right) = -4\pi G \rho \]

Now convert into dimensional quantities: \[ \rho(r) \equiv \rho(r) \left( \frac{Dn(r)}{\rho} \right)^{-n} \quad 0 < Dn < 1 \]

\[ \text{central density} \]
1) \[ \frac{n+1}{Pr} k \frac{1}{r^{2}} \frac{d}{dr} \left( r^{2} \rho c \left( \frac{1-n}{n} \right) \right) D_{n}^{m-1} \frac{1}{r^{2}} \frac{d}{dr} (r^{2} d\rho_{n}) = -4\pi \nu \rho c D_{n} \]

2) \[ \frac{(n+1) \rho c}{4\pi \nu} \frac{1}{r^{2}} \frac{d}{dr} (r^{2} d\rho_{n}) = -D_{n} \]

\[ \alpha = \frac{1}{r^{2}} \]

let \[ r^{2} = \xi \]

\[ \text{Dimensionless quantity} \]

\[ \frac{1}{\xi^{2}} \frac{d}{d\xi} \left( \xi^{2} \frac{d\rho_{n}}{d\xi} \right) \]

Lane-Emden eqn

(1870 Lane, Patent Office worker)

solving this allows us to write

\[ \rho(r) = \rho_{c} \left( \frac{D_{n} \left( \rho_{c} \xi \right)}{\rho_{c}} \right)^{\frac{n+1}{n}} \]

\[ \rho(r) = \rho_{c} \xi \]

\[ T(r) = \frac{\rho(r) \mu(r) M_{n}}{\rho(r) \xi} \]

if ideal gas

\[ \text{Boundary conditions:} \]

\[ \rho = \rho_{c} \]

\[ T = T_{c} \]

what do solutions to Lane-Emden eqn look like?

analytic solutions

\[ D_{0} = 1 - \frac{\epsilon_{0}^{2}}{\xi_{0}^{2}} \]

\[ D_{1} = \frac{\sin \xi_{1}}{\xi_{1}} \]

\[ D_{2} = \frac{\sin \xi_{2}}{\xi_{2}} \]

\[ D_{3} = \left( \frac{1 + \frac{\xi_{3}^{2}}{3} \right) \xi_{3} \]

all others must be solved numerically

1st zeroes:

\[ \xi_{0} \]

\[ \xi_{1} = \pi \]

\[ \xi_{5} = \infty \]
Surface defined @ \( \xi = \xi_1 \rightarrow p = 0 \Rightarrow Dn(\xi_1) = 0 \)

Center @ \( \xi = 0 \rightarrow \frac{dp}{dr} = -\frac{GMp}{r^2} - \frac{4\pi}{3} G \rho r^2 \rightarrow 0 \)

\[ \Rightarrow k \frac{n+1}{n} \frac{dp}{dr} = 0 \]

\[ \Rightarrow \frac{dp}{dr} = 0 \text{ @ } \xi = 0 \]

\[ \Rightarrow \frac{dDn}{d\xi} = 0 \text{ @ } \xi = 0 \]

What are the physical parameters?

radius: \( R = \ln \xi_1 \)

mass: \( M = \int_{\xi_0}^{\xi_1} 4\pi r^2 \rho(r) dr \)

\[ = 4\pi A_n \rho c \int_{\xi_0}^{\xi_1} \xi^2 \frac{Dn(\xi)}{d\xi} d\xi \]

\[ \Rightarrow \frac{d}{d\xi} \left[ \xi^2 \frac{dDn}{d\xi} \right] = -\xi^2 \frac{dP}{dr} \]

\[ \Rightarrow -4\pi A_n \rho c \xi_1 \frac{dDn}{d\xi} \bigg|_{\xi_1} \]

Some other relevant quantities:

\[ \phi = \frac{3}{4\pi} \frac{M}{n^2} = -\frac{3}{4\pi} \frac{4\pi A_n \rho c \xi_1^2}{dDn/d\xi} \]

\[ (R/l_1)^2 \]

\[ -3pc \left[ \frac{d\rho_c}{de_l} \right] \approx \text{average density of } p_c \]

\[ p_c = \frac{A_n^2 4\pi \rho c^2}{n+1} \]

\[ \approx \frac{4\pi c^2}{(n+1) \xi_1^2} \frac{1}{\left( \frac{g_0}{g_2} l_2 \right)^2} \approx \rho c^2 \frac{g_0}{\xi_1^2} \frac{m^2}{12k} \]

\[ \approx (4\pi (n+1) \frac{d\rho_c}{de_l})^2 \frac{1}{12k} \frac{g_0 M^2}{R^4} \]
Types & polytropes:

adiabatic gas:
\[ P = \frac{\rho}{\rho_0} \]
\[ \Rightarrow n = 1.5 \]

non-relativistic degenerate gas:
\[ P = \frac{(3\pi^2)^{2/3}}{5} \frac{k_b}{\hbar c} \left( \frac{\rho}{\mu m_{\text{eff}}} \right)^{5/3} \]
\[ \Rightarrow n = 1.5 \text{ also!} \]

relativistic degenerate gas:
\[ P = \frac{(3\pi^2)^{1/3}}{4} \hbar c \left( \frac{\rho}{\mu m_{\text{eff}}} \right)^{4/3} \]
\[ \Rightarrow n = 3 \text{ also!} \]

Fock-Dirac-Brillouin model:
\[ P = \frac{\rho k_B T}{\mu m_{\text{eff}}} + \beta \rho \]
\[ \rho_F = \frac{1}{3} \sigma a T^4 + (\rho - \rho_F) \]
\[ \Rightarrow P = \left( \frac{3(\rho - \rho_F)}{\rho} \right)^{1/3} \left( \frac{\rho}{\rho_0} \right)^{1/3} \]
\[ \Rightarrow n = 3 \text{ also!} \]