Rotational Energy and Angular Momentum Conservation

In this training set, you will learn how to use conservation of rotational energy and angular momentum to solve for the rotational motion of objects and systems:

\[ KE_r = \frac{1}{2} I_{COM} \omega^2 \quad L = \mathbf{r} \times \mathbf{p} = \mathbf{I}\omega \quad KE_r = \frac{L^2}{2I_{COM}} \]

Note that \( KE_r \) is a scalar quantity, and measures rotation about the center of mass; while \( L \) is a vector quantity pointing in the same direction as angular velocity. The change in rotational energy is caused by rotational work:

\[ \Delta KE_r = W_r = \mathbf{\tau} \cdot \Delta \theta \]

while the total energy of an object or system is a combination of translational and rotational kinetic and potential energies, which changes only when dissipative forces or torques act:

\[ \Delta E = \Delta(E_t + E_r) = \Delta(\frac{1}{2}mv^2 + PE_t + \frac{1}{2}I\omega^2 + PE_r) = W_d = F_d \cdot \Delta r + \mathbf{\tau}_d \cdot \Delta \theta \]

Changes in the angular momentum of a system occur only when net external torque acts on the system, causing an angular impulse:

\[ \Delta L = \Sigma \mathbf{\tau}_{ext} \Delta t \]

These quantities allow us to solve for the motion of systems by focusing on their initial and final states, and considering only changes caused by dissipative and/or external forces and torques. In many cases, the total energy, total momentum and/or total angular momentum of a system may be conserved.

Part 1: Translational and Rotational Energy and Work

We will apply a similar procedure as that practiced in Cycle 3 to examine both the translational and rotational energy of a system, considering cases where total energy may or may not be conserved. In part (c) you will be considering a torsional spring, which has a conservative torque law similar to the elastic force law for a spring:

\[ \mathbf{\tau} = -\kappa \Delta \theta \]

where \( \kappa \) is a constant with units of N-m.

Follow these steps to solve the following problems:

1. **Predict** both the rotational and translational motion of the object or system;
2. **Forces**: Identify the conservative forces or torques acting in the problem, and compute their corresponding potential energies;
3. **Energy Table**: Write down expressions for the translational and rotational kinetic and potential energies in the initial and final states of the motion; sum these to get total mechanical energies;
4. **Dissipation**: If dissipative forces are acting, compute the dissipative work;
5. **Constraint Equations**: Write down any constraint equations between rotational and translational motions;
6. **Work-Energy**: Write down the work energy theorem, \( \Delta E = E_f - E_i = W_d \), and solve for any unknown value(s).

The first example has been done for you.
(a) A 6.5 kg bowling ball (solid sphere, I_{COM} = \frac{2}{5}MR^2) of radius 10 cm is placed at the top of an 1.4 m high inclined ramp with incline angle \( \theta = 25^\circ \). Released from rest, the ball rolls down the ramp without slipping. What is the speed of the ball at the bottom of the ramp?

**Prediction of motion:**
The ball will gain both translational and rotational energy from its initial weight force potential energy.

**Conservative forces/torques and potential energy:**

Weight force: PE = mgh

**Energy Table:**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Initial</th>
<th>Final</th>
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</thead>
<tbody>
<tr>
<td>KE_t</td>
<td>0 J</td>
<td>( \frac{1}{2} mv^2 )</td>
</tr>
<tr>
<td>KE_r</td>
<td>0 J</td>
<td>( \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{2}{5} mR^2 \omega^2 )</td>
</tr>
<tr>
<td>PE_t</td>
<td>mgh = (6.5 kg)(9.81 m/s^2)(1.4 m) = 89.27 J</td>
<td>0 J</td>
</tr>
<tr>
<td>PE_r</td>
<td>0 J</td>
<td>0 J</td>
</tr>
<tr>
<td>Total E</td>
<td>89.27 J</td>
<td>( \frac{1}{2} mv^2 + \frac{1}{5} mR^2 \omega^2 )</td>
</tr>
</tbody>
</table>

**Dissipative forces/torque and work:**

None

**Constraint Equations:**

Rolling without slipping => \( v = R\omega \) throughout

**Work-energy theorem and solve for unknowns:**

\[
\Delta E = E_f - E_i = 0 \Rightarrow E_f = E_i
\]

\[
\Rightarrow 89.27 J = \frac{1}{2} mv^2 + \frac{1}{5} mR^2 \omega^2 = \frac{1}{2} mv^2 + \frac{1}{5} mv^2 = \frac{7}{10} mv^2
\]

\[
\Rightarrow v = ((10/7)(89.27 J)/(6.5 kg))^{1/2} = 4.4 \text{ m/s}
\]
(b) This problem involves dissipative work. A 2.0 kg metal bucket containing 5.5 kg of water is being slowly hoisted up from a well by a pulley wheel when the handle on the pulley wheel breaks. As the wheel unwinds, the bucket (initially at rest) starts to fall back down the well, and you attempt to slow it down by applying a constant friction force of 50 N at the top of the wheel. What is the speed of the bucket after it has fallen 5.0 m if the pulley wheel is a solid disk (I_{COM} = \frac{1}{2}MR^2) of mass 1.4 kg and radius 0.25 m?

Prediction of motion:

Conservative forces/torques and potential energy:

Energy Table:

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<tr>
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<tbody>
<tr>
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<tr>
<td>Total E</td>
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Dissipative forces/torque and work:

Constraint Equations:

Work-energy theorem and solve for unknowns:
(c) This problem involves rotational potential energy. Most wind-up toys have a torsional spring that is connected to a main rotating gear; by twisting up the spring and releasing it, the stored energy spins the gear and other parts of the toy. For this toy, the main gear wheel is a solid disk \( I_{\text{COM}} = \frac{1}{2} MR^2 \) with a mass of 300 grams and a radius of 3.0 cm (ignore the twist post), and the torsional spring has a constant \( \kappa = 4.5 \times 10^{-3} \, \text{N-m} \). If you wind the spring up by three revolutions and release it from rest, what is the maximum spin rate of the gear? Neglect friction in this mechanism.

**Prediction of motion:**

**Conservative forces/torques and potential energy:**

**Energy Table:**

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**Dissipative forces/torque and work:**

**Constraint Equations:**

**Work-energy theorem and solve for unknowns:**
**Part 2: Linear and Angular Momentum & Impulse**

We will apply a similar procedure to that practiced in Cycle 4 to compare the initial and final linear and angular momenta of a system, which is changed only by external forces and torques. Follow these steps to solve the following problems:

1. **Set up**: Predict the motion and define the system;
2. **Force diagram**: label only the external forces acting on the system, and define a coordinate system (including direction of positive rotation);
3. **Momentum Table**: Sketch the initial and final states and write down expressions for the linear and angular momenta;
4. **Impulses**: If external forces are present, compute the linear impulse; if these lead to a net torque, compute the angular impulse;
5. **Impulse-Momentum**: Write down the impulse-momentum theorems \( \Delta p = p_f - p_i = \Sigma F_{\text{ext}} \Delta t \) and \( \Delta L = L_f - L_i = \Sigma \tau_{\text{ext}} \Delta t \), and solve for any unknown value(s).

The first example has been done for you.

(a) A solid disk (I_{COM} = \frac{1}{2}MR^2) of mass 2.0 kg and radius 0.50 m is spinning clockwise at a rate of 0.50 radian/s on a flat, frictionless table. A second disk of mass 0.50 kg and radius 0.25 m, initially not spinning, is placed on top of the first disk. Friction causes the two disks to spin at the same rate. What is the final angular velocity of the system (magnitude and direction)?

**Prediction & System:**

System is two disks; the bottom disk will slow down while top disk spins up, until they have a common spin rate less than but in the same direction as the original rate.

**Force Diagram:**

Note: no linear motion

**Momentum Table:**

Sketch:

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<tr>
<th>Quantity</th>
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<tr>
<td>( L )</td>
<td>( \frac{1}{2}M\omega_1 + \frac{1}{2}M\omega_2 = \frac{1}{2}(2.0 \text{ kg})(0.50 \text{ m})^2(0.50 \text{ rad/s}) = 0.125 \text{ kg m}^2/\text{s} )</td>
<td>( \frac{1}{2}M\omega_1' + \frac{1}{2}M\omega_2' = (I_1+I_2)\omega' = \frac{1}{2}(2.0 \text{ kg})(0.50\text{m})^2 + \frac{1}{2}(0.50 \text{ kg})(0.25\text{m})^2\omega' = (0.266 \text{ kg m}^3)\omega' )</td>
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</table>

**External Impulses:**

None

**Impulse-Momentum Relation and Solve for Unknowns:**

\( \Delta L = L_f - L_i = 0 \Rightarrow 0.125 \text{ kg m}^2/\text{s} = (0.266 \text{ kg m}^3)\omega' \)

\( \Rightarrow \omega' = 0.47 \text{ rad/s pointing toward negative y} \)
(b) Similar to (a), only now a hoop (I_{COM} = MR^2) of mass 1.2 kg and radius 0.40 m, initially spinning counterclockwise at 1.0 radian/s, is placed on top of the solid disk. Again, friction causes the two disks to spin with the angular velocity. What is this angular velocity (magnitude and direction)?

Prediction & System:

Force Diagram:

Momentum Table:

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External Impulses:

Impulse-Momentum Relation and Solve for Unknowns:
(d) This problem involves both linear and angular momentum conservation: A 0.40 kg lump of clay is thrown at a speed of 1.3 m/s toward a 1.0 m ruler ($I_{\text{COM}} = 1/12 ML^2$) with mass 0.40 kg, initially at rest on a frictionless table. The clay sticks to one end of the ruler, and the ruler+clay system starts to slide and spin about its center of mass. Where is the center of mass located, and what are the final linear and angular velocities (magnitude and direction) of the system? You can treat the lump of clay as a point mass.

Prediction & System:

Force Diagram:

Momentum Table:
Sketch:

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External Impulses:

Impulse-Momentum Relation and Solve for Unknowns:
(d) This problem involves both linear and angular impulses: A 0.25 kg cue ball of radius 2.8 cm \((I_{COM} = \frac{2}{5} MR^2)\) is struck in such a way as to give it an initial speed of 2.5 m/s toward the right. Friction acts between the ball and table surfaces, and eventually the ball rolls without slipping. What are the final linear and angular velocities (magnitude and direction) of the ball?

**Prediction & System:**

**Force Diagram:**

**Momentum Table:**

Sketch:

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**External Impulses:**

**Impulse-Momentum Relation and Solve for Unknowns:**