Physics 223: Stellar Astrophysics
Homework #3
Due Friday November 4th at 5pm in box in front of SERF 340

Reading: HKT Chapters 4-6
Exercises: [120 pts total]

(1) Relative velocity distribution of two Maxwell-Boltzmann populations [10 pts]

In lecture we have calculated the weighted average $<\sigma v>$ for nuclear reaction rates by assuming that the relative velocity of the two fusing nuclei has a Maxwell-Boltzmann distribution. Here, you will prove that this assumption was justified.

If the velocities of a set of identical, distinguishable particles with mass $m$ has a Maxwell-Boltzmann distribution, then the fraction of particles that have velocity in the range $[v, v + dv]$ is given by

$$f(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} dv$$

Suppose there are two different sets of particles, with masses $m_1$ and $m_2$, with Maxwellian velocity distributions centered on $v_1$ and $v_2$, respectively. Show that the distribution of relative velocities $v = v_1 - v_2$ is also Maxwellian (you will find it useful to rewrite $v_1$ and $v_2$ in terms of the center of mass and relative velocities).

(2) Nuclear reaction chains and abundances [30 pts]

Reaction rates for nuclear reactions can be written as:

$$r_{ij} = (1 + \delta_{ij})^{-1}\langle \sigma v \rangle_{ij} n_i n_j$$

where $n_i$ is the number density of species $i$, satisfying
Consider the PPI chain:

1. \( ^1\text{H} + ^1\text{H} \rightarrow ^2\text{H} + \text{e}^+ + \nu_e \)
2. \( ^2\text{H} + ^1\text{H} \rightarrow ^3\text{He} + \gamma \)
3. \( ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2 ^1\text{H} \)

(1) [15 pts] Turn this reaction chain into a series of differential equations for the abundances of \(^1\text{H}, ^2\text{H}, ^3\text{He}\) and \(^4\text{He}\) over time (i.e., \(dn_{^3\text{He}}/dt = ...\)) based on reaction constants \(\lambda_x = <\sigma v>_x\) for \(x = 1, 2, 3\) reactions listed above.

(2) [15 pts] Assume that the abundance of \(^2\text{H}\) is an equilibrium. Using the Gaussian approximation for the Gamov peak (Eqn 6.42 in HKT; remember that \(\mu = \) is the reduced atomic mass \(A_1A_2/(A_1+A_2)\) and assuming values of \(S_1 = 3.8 \times 10^{-22}\) keV-barns, \(S_2 = 2.5 \times 10^{-4}\) kev-barns and a temperature of \(1.5 \times 10^7\) K, determine the equilibrium abundance of \(^2\text{H}\) relative to \(^1\text{H}\) and compare to the primordial abundance of \(2.8 \times 10^{-5}\) from Kirkman et al. (2003, ApJS, 149, 1).

(3) Helium fusion in degenerate conditions (He flash) [20 pts]

Suppose you have 1 gram of pure \(^4\text{He}\) in the center of a pre–helium flash red giant. The density and temperature of the gram are, respectively, \(\rho = 2 \times 10^5\) g cm\(^{-3}\) and \(T = 1.5 \times 10^8\) K. This is hot enough to burn helium by the triple-alpha reaction, which is the only reaction we will consider here. The energy generation rate for this reaction is given by

\[
\epsilon_{3\alpha} = \frac{5.1 \times 10^8 \rho^2 Y^3}{T_9^3} e^{-4.4027/T_9} \text{ erg g}^{-1} \text{ s}^{-1}
\]

where \(Y\) is the helium mass fraction and \(T_9\) is the temperature in units of \(10^9\) K.
Follow the time evolution of this gram as helium burning proceeds, by computing the temperature $T(t)$ as a function of time. Start the clock running at $t = 0$ with the stated conditions. For simplicity, assume that the density and the helium concentration remain constant for all time and that no heat is allowed to leave the gram. Compute $T(t)$ numerically until such time that the material begins to become nondegenerate, using the demarcation line $\rho \approx 10^{-8} T^{3/2} \text{g/cm}^3$ (densities below this will be nondegenerate).

For a given input of heat energy $\Delta E$, you can calculate the change in temperature of a mass $\Delta m$ from the specific heat $c_V$ using

$$\Delta T = \left( \frac{\Delta E}{\Delta m} \right) \frac{1}{c_V}$$

Note that you will need to use the total specific heat $c_V = c_{V,e} + c_{V,\text{He}}$. The degenerate electron specific heat is given by

$$c_{V,e} = \frac{1.35 \times 10^5}{\rho} T x (1 + x^2)^{1/2} \text{ erg g}^{-1} \text{ K}^{-1}$$

where $x = p_F/mc$ is the normalized Fermi momentum and $\rho \approx 2 \times 10^6 x^3 \text{ g/cm}^3$. Convince yourself that this term dominates in the degenerate regime. The specific heat for an ideal gas of helium ions is

$$c_{V,\text{He}} = \frac{3k}{4m_p}$$

Plot temperature $T(t)$ versus time in days. You will recognize the He flash when it happens because the temperature will suddenly skyrocket. Indicate whether the result you get is reasonable and what criteria you use to make that determination.

(4) **Nuclear statistical equilibrium** [30 pts]

In the normal course of evolution of a massive star, the end products of nuclear burning are elements in the iron region of nucleon number. If the
temperatures get high enough, the radiation field is capable of initiating photodisintegration, and all the iron-peak elements end up as individual nucleons. This can happen on such rapid time scales that the abundances of nuclei (as functions of temperature and density) can be calculated approximately as if the gas were in chemical equilibrium using a version of the Saha equation.

To look at this in a very simplified way, consider a gas composed only of $^{56}\text{Ni}$ and $^4\text{He}$ where the “chemical reaction” between them is

$$^{14}^4\text{He} \leftrightarrow ^{56}\text{Ni} + Q$$

The Q-value can be determined from the mass excesses (mass above A times the proton mass),

$$(M - A_m_p)c^2 = 2.42 \text{ MeV for } ^4\text{He}$$

$$(M - A_m_p)c^2 = -53.9 \text{ MeV for } ^{56}\text{Ni}$$

In equilibrium, the chemical potentials obey the relation

$$^{14}\mu_{\text{He}} = \mu_{\text{Ni}}.$$ 

Since the nuclei are non-degenerate, the number density of species $i$ (either He or Ni) obey the Maxwell-Boltzmann distribution,

$$n_i = g_i \left( \frac{m_i k T}{2\pi \hbar^2} \right)^{3/2} e^{(\mu_i - m_i c^2)/k T}$$

where $\mu_i - m_i c^2 = E_i$ is the particle energy.

(a) [15 pts] Derive an equivalent of the Saha equation for this reaction, pretending that you are dealing with atoms and ions and assuming that both nuclei are in their ground states. To do this, substitute the Maxwell-Boltzmann relation for the number densities into the equilibrium relation for the chemical potentials, and derive an equation whose left hand side is an appropriate ratio of number densities. Assume the degeneracy values $g_i$ are equal to 1 (which is appropriate since the ground state spins are zero).
Note that there will be two differences from the Saha equation you encountered earlier in the term: First, there are only two species so there will only be two different n’s. Second, the multiple He nuclei in the reaction will result in a nonlinear dependence on n_{He}. The Q-value plays the role of the ionization energy.

(b) [5 pts] Re-cast your Saha equation so that the unknowns are the mass fractions X_4 and X_{56} where X_4 + X_{56} = 1.

(c) [10 pts] Fix the density to be ρ = 10^7 g cm^{-3} and solve for X_4 and X_{56} for temperatures in the range 4.5 < T_9 < 6.5, where T_9 is the temperature in units of 10^9 K. Plot your results for the mass fractions versus temperature. At what temperature is X_4 = X_{56}?

(5) MESA Exercise: Zero Metallicity Stars [30 pts]

(this problem is from the UC Berkeley Stellar Astrophysics Grad Class)

Consider a primordial massive star (M ∼ 30M_☉) formed early in the Universe that initially has no metals but has both hydrogen (X ≈ 0.75) and helium (Y ≈ 0.25).

Carbon and Oxygen are products of He fusion. Thus, once He fusion has occurred for some amount of time, there will be enough C & O around for H fusion to proceed via the CNO cycle. This means that primordial massive stars spend the vast majority of their lives fusing H → He via the CNO cycle (because they are so hot), using the trace amount of CNO generated by a brief epoch of He fusion!

You’re going to compare the evolution of a 30 M_☉ star with both zero initial metals and an initial solar mix, looking at how the CNO cycle modulates the early evolution of these stars.

(a) [10 pts] Use MESA to evolve two 30M_☉ stars; one with Z = 0 and one with solar Z (see HW 1). Plot their tracks on the HR diagram, their internal temperature-density profiles, their abundances and their power generation. Report these plots for both stars at the following benchmarks: (1) when they reach the Zero Age Main Sequence, (2) when
they deplete core Hydrogen (end of Main Sequence), (3) when they
deplete core Helium (end of He-burning main sequence) and (4) when
they deplete core Carbon (end of C-burning main sequence)

(b) [5 pts] At what ages do the two stars reach these benchmarks? What
are the core temperatures and densities of the stars at these
benchmarks?

(c) [5 pts] What is the relative contribution of CNO fusion at the start
(just after ZAMS) and end of the Main Sequence (just before H
depletion) for these stars?

(d) [5 pts] How do the core temperatures differ between these stars at
ZAMS? Why might this be?

(e) [5 pts] What is the value of the CNO mass fraction \( Z_{\text{CNO}} = Z_C + Z_N + Z_O \)
required for CNO fusion to dominate in the metal-poor star? Roughly
how long does it take that star to generate this amount of CNO?