(1) Relative velocity distribution of two Maxwell-Boltzmann populations [10 pts]

We want to solve for the distribution in the relative velocity, which is combination of the individual velocities:

\[ f(\vec{v}_r) \, d\vec{v}_r = f(\vec{v}_1) f(\vec{v}_2) \, d\vec{v}_1 \, d\vec{v}_2 \]

\[ = \left( \frac{m_1}{2\pi kT} \right)^{3/2} \left( \frac{m_2}{2\pi kT} \right)^{3/2} e^{-\left( \frac{m_1 v_1^2}{2kT} + \frac{m_2 v_2^2}{2kT} \right)} \, d\vec{v}_1 \, d\vec{v}_2 \]

Replace the individual velocity terms with the relative and center of mass velocities:

\[ v_r = v_1 - v_2 \]

\[ v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Squaring these with an appropriate constant in front:

\[ (m_1 + m_2) v_c^2 = \frac{m_1 v_1^2 + m_2 v_2^2 + 2m_1 m_2 v_1 v_2}{m_1 + m_2} \]

\[ \frac{m_1 m_2}{m_1 + m_2} v_r^2 = \frac{m_1 m_2 v_1^2 + m_1 m_2 v_2^2 - 2m_1 m_2 v_1 v_2}{m_1 + m_2} \]

yields:

\[ (m_1 + m_2) v_c^2 + \frac{m_1 m_2}{m_1 + m_2} v_r^2 = m_1^2 v_1^2 + m_2^2 v_2^2 \]
We can convert the differential elements using the Jacobian determinant:

\[
dv_1 dv_2 = \begin{vmatrix} \frac{dv_1}{dv_c} & \frac{dv_1}{dv_r} \\ \frac{dv_2}{dv_c} & \frac{dv_2}{dv_r} \end{vmatrix} dv_c dv_r
\]

The derivatives come from solving for \(v_1\) and \(v_2\) in terms of \(v_c\) and \(v_r\):

\[
v_1 = v_c + \frac{m_2}{m_1 + m_2} v_r \quad v_2 = v_c - \frac{m_1}{m_1 + m_2} v_r
\]

\[
\Rightarrow dv_1 dv_2 = \begin{vmatrix} \frac{1}{m_1 + m_2} & -\frac{1}{m_1 + m_2} \end{vmatrix} dv_c dv_r = dv_c dv_r
\]

Combining these:

\[
f(v_r) d^3 v_r = \left( \frac{m_1}{2\pi kT} \right)^{3/2} \left( \frac{m_2}{2\pi kT} \right)^{3/2} e^{-\frac{(m_1 + m_2)v_r^2}{2kT}} e^{-\left(\frac{\mu v_r^2}{2kT}\right)} d^3 v_r
\]

where we’ve explicitly inserted the reduced mass:

\[
\mu = \frac{m_1 m_2}{m_1 + m_2}
\]

We can integrate over \(v_c\) since this variable doesn’t appear on the left:

\[
f(v_r) d^3 v_r = \left( \frac{m_1}{2\pi kT} \right)^{3/2} \left( \frac{m_2}{2\pi kT} \right)^{3/2} \left( \frac{2\pi kT}{m_1 + m_2} \right)^{3/2} e^{-\left(\frac{\mu v_r^2}{2kT}\right)} d^3 v_r
\]

\[
= \left( \frac{\mu}{2\pi kT} \right)^{3/2} e^{-\left(\frac{\mu v_r^2}{2kT}\right)} d^3 v_r
\]

Thus, the relative velocity distribution is Maxwellian with the mass converted to the reduced mass.

(2) Nuclear reaction chains and abundances [30 pts]
(a) [15 pts] For each element, consider the reactions that create and destroy that element as sources and sinks. Remembering that the reaction rate:

\[ r_i = \frac{n_x n_y \langle \sigma v_{xy} \rangle}{1 + \delta_{xy}} = \frac{n_x n_y \lambda_i}{1 + \delta_{xy}} \]

will either remove reactants (-dn/dt) or add products (+dn/dt), and accounting for species identity, we can write the time derivatives of the abundances are:

\[ \frac{dn_H}{dt} = -2 \frac{\lambda_1 n_H^2}{2} - \lambda_2 n_H n_D + 2 \frac{\lambda_3 n_{3He}^2}{2} \]

\[ \frac{dn_D}{dt} = -\lambda_2 n_H n_D + \frac{1}{2} \lambda_1 n_H^2 \]

\[ \frac{dn_{3He}}{dt} = -2 \frac{\lambda_3 n_{3He}^2}{2} - \lambda_2 n_H n_D \]

\[ \frac{dn_{4He}}{dt} = \frac{1}{2} \lambda_3 n_{3He}^2 \]

(b) [15 pts] If n_D is constant:

\[ \lambda_2 n_H n_D = \frac{1}{2} \lambda_1 n_H^2 \]

\[ \Rightarrow \frac{n_D}{n_H} = \frac{1}{2} \frac{\lambda_1}{\lambda_2} \]

The expression for the reaction rate \( <\sigma v> \) from HKT:

\[ <\sigma v> = \lambda = \frac{0.72 \times 10^{-18} S \alpha^2}{\mu Z_\alpha Z_X} e^{-aT_6^{-1/3}} \frac{T_6^{2/3}}{cm^3/s} \]

where \( \alpha \) is the target particle and X is the incident particle, Z is the proton number, \( \mu \) is the reduced mass of the reaction, \( T_6 \) is the temperature in \( 10^6 \) K = 15, S is in keV-barns and
\[ a = 42.49 \left( Z_\alpha^2 Z_X^2 \mu \right)^{1/3} \]

The ratio of the reactions can be written as

\[
\frac{\lambda_1}{\lambda_2} = \frac{\left( \mu Z_\alpha Z_X \right)_2 S_1 a_1^2}{\left( \mu Z_\alpha Z_X \right)_1 S_2 a_2^2} e^{-(a_1-a_2)T_6^{-1/3}}
\]

For the first reaction: \( Z_\alpha = Z_X = 1, \mu = 0.5 \) and \( S = 3.8 \times 10^{-22} \) keV-bands, so \( a_1 = 33.72 \); for the second reaction: \( Z_\alpha = Z_X = 1, \mu = 2/3 \) and \( S = 2.5 \times 10^{-4} \) keV-bands, so \( a_2 = 37.12 \). The ratio is then:

\[ n_D/n_H = 0.5 \lambda_1/\lambda_2 = 3.3 \times 10^{-18} \]

This is vanishingly small compared to the primordial abundance!

(3) **Helium fusion in degenerate conditions (He flash)** [20 pts]

Here’s a few plots tracking the temperature, energy generation rate, specific heat and degeneracy parameter \( (2.8 \times 10^{-6} \times \rho/T^{1.5} > 1 \) for degenerate, < 1 for nondegenerate) based on the code posted on the course website; the flash occurs about 8 days into the simulation.
Is this reasonable? The rapid increase in energy generation occurs because the temperature moves into the Gamov peak for He fusion, which releases heat; because the electrons are degenerate, they can largely absorb this heat (low heat capacity). But as the degeneracy lifts, the specific heat increases sharply. We run into a feedback mechanism – increased temperature leading to increased energy production, which causes the electrons to lose their degeneracy and become hotter, which feeds higher He fusion rates. A flash!

(4) Nuclear statistical equilibrium [30 pts]

(a) [15 pts] First we compute the Q value:

\[ Q = \Delta mc^2 = 14(m_{\text{He}}-4m_p)c^2 - (m_{\text{Ni}} - 56m_p)c^2 \]

\[ = 14m_{\text{He}}c^2 - m_{\text{Ni}}c^2 = -87.78 \text{ MeV} \]

We can solve for \( \mu \) from the Boltzmann equation, with \( E_0 = mc^2 \) (note this is a little sloppy in terms of units):

\[ \mu_i = m_i c^2 + kT \ln \left( \frac{n_i}{g_i} \right) + \frac{3}{2} kT \ln \left( \frac{2\pi \hbar^2}{m_i kT} \right) \]

So the chemical equilibrium equation with \( g = 1 \) becomes

\[ 14\mu_{\text{He}} - \mu_{\text{Ni}} = \Delta \left( m_i c^2 \right) + kT \ln \left( \frac{n_{\text{He}}^{14}}{n_{\text{Ni}}} \right) + \frac{3}{2} kT \ln \left( \frac{m_{\text{Ni}}}{m_{\text{He}}^{14}} \right) + 13\frac{3}{2} kT \ln \left( \frac{2\pi \hbar^2}{kT} \right) = 0 \]

and hence:
\[
\frac{n_{Ni}}{n_{He}} = \left( \frac{m_{Ni}}{m_{He}} \right)^{3/2} \left( \frac{2\pi\hbar^2}{kT} \right)^{39/2} e^{-Q/kT}
\]

(convince yourself the units are now correct here).

(b) [5 pts] Substituting in the mass fraction:

\[
X_i = \frac{m_in_i}{\rho}
\]

and the atomic mass \(A_i = m_i/m_p\) into the equation above and redistributing the proton mass term \(m_p\):

\[
\frac{X_{56}}{X_{4}^{14}} = \frac{X_{56}}{(1 - X_{56})^{14}} = \left( \frac{\rho}{m_p} \right)^{13} \left( \frac{A_{Ni}}{A_{He}^{14}} \right)^{5/2} \left( \frac{2\pi\hbar^2}{m_pc^2kT} \right)^{39/2} e^{-Q/kT}
\]

(c) [10 pts] First lets put the above into numeric form using the values we know in units of \(T_9\):

- \(kT = 8.6\times10^{-11}\) MeV/K \(\times 10^9\) K = 0.086 MeV \(T_9\)
- \(Q/kT = -87.78\) MeV/0.086 MeV \(T_9 = -1020/T_9\) (note the sign!)
- \(2\pi\hbar^2/m_pc^2kT = 2\pi \left[ 2\times10^{-11}\right.\) MeV-cm\(^2/938\) MeV \(\times 0.086\) MeV \(T_9\) = 3.1\times10^{-23}\) cm\(^2\) \(T_9^{-1}\)
- \(A_{Ni}/A_{He}^{14} = 2.1\times10^{-7}\)

Combining these values with \(\rho/m_p = \rho N_A = 10^7\) g/cm\(^3\) \(\times 6.02\times10^{23}\) g\(^{-1}\) = 6.02\times10^{30}\) cm\(^{-3}\):

\[X_{56}/X_{4}^{14} = 3.3\times10^{-56} T_9^{-19.5} 10^{443/T_9}\]

or

\[\log(X_{56}/X_{4}^{14}) = -55.48 - 19.5\log(T_9) + 443/T_9\]

[note conversion of e\(^x\) to 10\(^x\) for easier calculation]

Let’s try \(T_9 = 5.0\):

\[\log(X_{56}/X_{4}^{14}) = -55.48 - 19.5 (0.699) + 88.6\]
\[ X_{56}/X_4^{14} = (1-X_4)/X_4^{14} = 3.0 \times 10^{19} \]

\[ = 1 - X_4 - 3.0 \times 10^{19} X_4^{14} = 0 \]

Numerically, we can find that \( X_4 = 0.0405 \Rightarrow X_{56} = 1 - X_4 = 0.9595 \): in favor of Nickel.

Using a simple code (included with solution), we can generate a run of \( X_4 \) and \( X_{56} \) versus temperature:

\[
X_4 = X_{56} \text{ at around } 6 \times 10^9 \text{ K.}
\]

**5) MESA Exercise: Zero Metallicity Stars [30 pts]**

(a) [10 pts] See the inlists attached to the solutions; the only change between the zero metallicity and solar metallicity models was to set initial \( z = 0.0 \) and initial \( y = 0.25 \) in \&controls. Note that the metal-poor star does not converge to C-burning limit as the timestep falls below \( 10^{-6} \) s (microsecond sampling for a 5 million-year old star!), Plots are as follows:
Metal-poor star @ ZAMS:
Solar-metal star @ ZAMS:
Metal-poor star @ end of MS:
Solar-metallicity star @ end of MS:
Metal-poor star @ end of He-burning:

![Graphs and diagrams showing properties of metal-poor stars at the end of He-burning process.](image-url)
Solar-metallicity star @ end of He-burning:
Solar-metallicity star @ end of C-burning:

- [Graph 1:log L vs log T]
- [Graph 2:log Temperature vs log Density]
- [Graph 3:Abundance vs log mass fraction]
- [Graph 4:log ergs/s vs log P]
(b) [5 pts]: Here’s a table of my results

<table>
<thead>
<tr>
<th>Metallicity</th>
<th>$X = 0.75$, $Y = 0.25$</th>
<th>Solar metallicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age to ZAMS (yr)</td>
<td>$5.53 \times 10^4$</td>
<td>$2.75 \times 10^4$</td>
</tr>
<tr>
<td>Age to end of Main</td>
<td>$5.24 \times 10^6$</td>
<td>$5.21 \times 10^6$</td>
</tr>
<tr>
<td>Sequence (yr)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Age to end of He-</td>
<td>$5.65 \times 10^6$</td>
<td>$5.75 \times 10^6$</td>
</tr>
<tr>
<td>burning (yr)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Age to end of C-</td>
<td>N/A</td>
<td>$5.77 \times 10^6$</td>
</tr>
<tr>
<td>burning (yr)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ZAMS log $T_c$ &amp; log $\rho_c$</td>
<td>$8.08, 2.08$</td>
<td>$7.57, 0.54$</td>
</tr>
<tr>
<td>End of MS log $T_c$ &amp; log $\rho_c$</td>
<td>$8.13, 2.15$</td>
<td>$7.79, 1.12$</td>
</tr>
<tr>
<td>End of He-burn log $T_c$ &amp; log $\rho_c$</td>
<td>$8.50, 3.33$</td>
<td>$8.50, 3.35$</td>
</tr>
<tr>
<td>End of C-burn log $T_c$ &amp; log $\rho_c$</td>
<td>N/A</td>
<td>$9.08, 6.01$</td>
</tr>
</tbody>
</table>

(c) For metal-poor star, CNO is about equivalent to pp, while for solar-z stars CNO is about 2.5 orders of magnitude stronger. This makes sense since there is very little C, N or O in the metal-poor star (abundances are very low), which is mainly supplied by the small rate of He fusion that is present even at ZAMS.

(d) The core temperature of the metal-poor star is always greater (by a factor of $\approx 3$) then the solar-Z star. This is due to the reduced efficiency of the CNO cycle, which requires the star to contract and drive up its temperature to match its outward luminosity (about the same between the stars). Note how the metal-poor star core density is higher, and
radius smaller (higher $T_{\text{eff}}$), than the solar-metallicity star.

(e) In my calculations, CNO fusion $\approx$ pp fusion at model number 1140 (age $= 4.9 \times 10^4$ yr); at this point the core metallicity $= 2.58 \times 10^{-11}$ – tiny! But not zero, and enough to facilitate the CNO cycle at high temperatures which is far more efficient than pp.