Physics 223: Stellar Astrophysics
Homework #4
Due Friday November 18th at 5pm in box in front of SERF 340

Reading: HKT Chapters 2, 3.4, 4.1-4.8

Exercises: [200 pts total]

(1) Photon walk [50 pts]

(a) [10 pt] This can be deduced with a little deductive reasoning. First, note that the first step will always have length $l$. The second step will have length:

$$|\Delta_2| = |\vec{r}_1 + \vec{r}_2| = \sqrt{l^2 + l^2 + 2l \cos \theta_{12}} = \sqrt{2l} \sqrt{1 + \cos \theta_{12}}$$

where $\theta_{12}$ is the angle between the two vectors. If we repeat this step many many times, on average we would find $<\cos \theta_{12}> = 0$ and hence

$$<|\Delta_2|> = \sqrt{2l}$$

Let’s proceed to the third step, with $\theta_{23}$ being the angle between the sum of the first two steps and the third step:

$$|\Delta_3| = |\vec{r}_1 + \vec{r}_2 + \vec{r}_3| = \sqrt{2l^2 (1 + \cos \theta_{12}) + l^2 + \sqrt{2l^2} \sqrt{1 + \cos \theta_{12} \cos \theta_{23}}}$$

$$= \sqrt{3l} \sqrt{1 + \frac{2}{3} \cos \theta_{12} + \frac{2\sqrt{2}}{3} \sqrt{1 + \cos \theta_{12} \cos \theta_{23}}}$$

Again, if we were to repeat this two-step process many many times, we would find that the large square root would be on average 1 (since $<\cos \theta_{12}> = 0$ and $<\cos \theta_{23}> = 0$), so that:

$$<|\Delta_3|> = \sqrt{3l}$$

More generally, at each step:

$$|\Delta_n| = \sqrt{|\Delta_{n-1}|^2 + l^2 + 2|\Delta_{n-1}| l \cos \theta_{n,n-1}}$$

which on average yields
\[ \langle |\Delta_n| \rangle = \sqrt{\langle |\Delta_{n-1}|^2 + l^2} \]

since the \( \cos \theta \) term will average out over many steps. Then we can just reproduce the sequence:

\[ \langle |\Delta_1| \rangle = l \]
\[ \langle |\Delta_2| \rangle = \sqrt{l^2 + l^2} = \sqrt{2}l \]
\[ \langle |\Delta_3| \rangle = \sqrt{2l^2 + l^2} = \sqrt{3}l \]

... 

\[ \langle |\Delta_N| \rangle = \sqrt{(N - 1)l^2 + l^2} = \sqrt{N}l \]

If we want to travel a distance

\[ R = \sqrt{N}l \]

this would indeed require \( N \) steps where

\[ N = (R/l)^2 \]

The distance travelled is

\[ d = Nl = R^2/l \]

(b) [10 pt] See the online solutions for an example python notebook to do this simulation. 10,000 runs takes some time, but is necessary for sufficient statistics. My simulation (using the angle draw as described in the assignment) gave:

\[ \text{min steps} = 343 \]
\[ \text{max steps} = 16129 \]
\[ \text{median steps} = 2090.00 \]
\[ \text{mean steps} = 2525.284 \]

The mean and median straddle the expected value of \( 50^2 = 2500 \)
The distribution of values is shown above; note that this has the classic rapid rise and slow decay that we often associate with flares and supernovae explosions. While the decay of a supernova is generally powered by the radioactive decay of elements, some features of the light curve can be explained by the random walk associated with radiation coming out of an opaque medium.

I also plot above the distribution of outward points just to make sure we were spherically symmetric.
(c) [10 pt] This is a classic collision cross-section argument we used previous for nuclear collisions.

Consider a cylinder with length $l$ and cross-section area $A$ (see diagram) through which photons are traveling (we can assume they all travel in one direction since we’re looking to see how the flow of energy drops off in a given direction). Inside the tube there are $N_e = n_e \times \text{Volume} = n_e A$ electrons that are relatively stationary compared to the photons, which with interaction cross section $\sigma_T$. Then the probability of a “collision” between a photon and an electron is:

$$P(\text{collision}) = \frac{\text{Area of colliders}}{\text{Area of cylinder}} = \frac{N_e \sigma_T}{A} = n_e l A \frac{\sigma_T}{A} = n_e l \sigma_T$$

This collision probability $\approx 1$ when

$$l = \frac{1}{n_e \sigma_T}$$

but since $n_e = \rho / \mu_e m_p$

$$l = \frac{\mu_e m_p}{\rho \sigma_T}$$

For the Sun, these yield values of:

<table>
<thead>
<tr>
<th>Region</th>
<th>$\rho$ (g/cm$^3$)</th>
<th>$\mu_e \approx 2/(1+x)$</th>
<th>$l$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>150</td>
<td>1.14</td>
<td>0.019</td>
</tr>
</tbody>
</table>
For the total path, let’s take the average density \(<\rho> = 1.4 \text{ g/cm}^3\) to compute an average \(<l> = 2 \text{ cm}\); the total path length as:

\[d \approx R^2/\langle l \rangle = (7 \times 10^{10} \text{ cm})^2/(2 \text{ cm}) = 2.5 \times 10^{21} \text{ cm}\]

and neglecting the scattering time itself:

\[t = d/c = (2.5 \times 10^{21} \text{ cm})/(3 \times 10^{10} \text{ cm/s}) = 8 \times 10^{10} \text{ s} \approx 3000 \text{ years}\]

With a range of 0.4 - 3 \times 10^5 \text{ yr} based on the length scales above

(d) [20 pt] See the attached plots for the density and pathlength as a function of radius, where the polytropic radial coordinate and density variables have been scaled to:

\[r = r_3 \xi\]

\[r_3 = \sqrt{\frac{P_c}{\pi G \rho c}} = 8.7 \times 10^{10} \text{ cm}\]

\[\rho = \rho_c \theta^3 = 146 \theta^3 \text{ g/cm}^3\]

Using the same value of \(\mu_e\) as above gives:

\[l = \frac{\mu_e m_p}{\rho \sigma_T} = 1.9 \times 10^{-2} \theta^{-3} \text{ cm}\]

Note we reproduce the core pathlength from before when \(\theta = 1\).
(2) **Adiabatic gradient for combined ideal and photon gases.** [50 pt]

(a) [10 pt] For constant pressure, start from the radiation pressure term which depends only on temperature:

\[ \partial P_{rad} = -\partial \beta P = \frac{4}{3} a T^3 \partial T \]

\[ \Rightarrow \left( \frac{\partial \beta}{\partial T} \right)_P = -\frac{4}{T} \frac{P_{rad}}{P} = -\frac{4}{T}(1 - \beta) \]

For constant temperature, the radiation pressure is constant, hence:

\[ \partial P = \partial P_{gas} = P \partial \beta + \beta \partial P \]

\[ \Rightarrow \left( \frac{\partial \beta}{\partial P} \right)_T = \frac{1}{P}(1 - \beta) \]

(b) [10 pt] At constant total pressure:

\[ \partial P = 0 = \partial P_{gas} + \partial P_{rad} \]

\[ = \frac{\partial \rho}{\rho} P_{gas} + \frac{\partial T}{T} P_{gas} + \frac{4 \partial T}{T} P_{rad} \]

\[ = (\partial \ln \rho) \beta P + (\partial \ln T) \beta P + 4(\partial \ln T)(1 - \beta)P \]

Cancelling the common pressure term and rearranging gives

\[- \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P \frac{4 - 3\beta}{\beta} \]

(c) [10 pt] The specific heat is the change in heat energy per gram of material for a change in temperature of 1°C at constant pressure:

\[ c_P = \left( \frac{\partial q^*}{\partial T} \right)_P \]

where “*” signifies per unit mass. We can relate this to the first law of thermodynamics:

\[ \partial Q = \partial U + P \partial V \]
where \(dQ\) is the heat added, \(dU\) the change in internal energy and \(PdV\) the work done by the gas. Rewriting this to be per unit mass, and noting that the volume per unit mass is just \(1/\rho\):

\[
\partial q^* = \partial u^* + P\partial(1/\rho) = \partial u^* - \frac{P}{\rho^2} \partial \rho
\]

Taking the temperature derivative of both sides, note that the second term can be related back to the thermal density gradient:

\[
- \frac{P}{\rho^2} \frac{\partial \rho}{\partial T} = - \frac{P}{T \rho} \frac{\partial \ln \rho}{\partial \ln T} = \frac{P}{T \rho} \delta
\]

And since \(P_{\text{gas}} = \beta P = \rho kT/\mu m_H\),

\[
- \frac{P}{\rho^2} \frac{\partial \rho}{\partial T} = \frac{k}{\mu m_H} \frac{4 - 3\beta}{\beta^2}
\]

For the first term, note that the energy per unit mass is divided between particles and photons. Assuming the former are monoatomic (ionized) and recalling the per particle energy is \(3/2 \, kT\) for an ideal gas, the energy density per mass is:

\[
u^* = \frac{3}{2} \frac{kT}{\bar{m}} + \frac{aT^4}{\rho} = \frac{3}{2} \frac{kTn}{\rho} + \frac{aT^4}{\rho}
\]

\[
= \frac{1}{\rho} \left( \frac{3}{2} P_{\text{gas}} + 3P_{\text{rad}} \right) = \frac{3P}{\rho} (1 - \frac{1}{2} \beta)
\]

Therefore:

\[
\frac{\partial u^*}{\partial T} = - \frac{3P}{\rho T} (1 - \frac{1}{2} \beta) \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P - \frac{3P}{2\rho} \left( \frac{\partial \beta}{\partial T} \right)_P
\]

\[
= \frac{3k}{\beta \mu m_H} (1 - \frac{1}{2} \beta) \left( \frac{4 - 3\beta}{\beta} \right) + \frac{6k}{\beta \mu m_H} (1 - \beta)
\]

Putting this all together:

\[
c_P = \left( \frac{\partial q^*}{\partial T} \right)_P = \frac{k}{\beta \mu m_H} \left( 3(1 - \frac{1}{2} \beta) \left( \frac{4 - 3\beta}{\beta} \right) + 6(1 - \beta) + \frac{4 - 3\beta}{\beta} \right)
\]
After some manipulation of the first two terms in the long parenthetical expression you will get

\[ c_P = \frac{k}{\mu m_H} \left[ \frac{3}{2} + \frac{3(4 + \beta)(1 - \beta)}{\beta^2} + \frac{4 - 3\beta}{\beta^2} \right] \]

(d) [10 pt] The lead term for the adiabatic index \( \frac{P}{T\rho} = \frac{k}{\beta \mu m_H} \), so this expression expands to:

\[ \nabla_{ad} = \frac{P}{T\rho c_P} \delta = (4 - 3\beta) \left[ \frac{3}{2} \beta^2 + 3(4 + \beta)(1 - \beta) + (4 - 3\beta) \right]^{-1} \]

It’s simply algebraic manipulation to get to the final expression:

\[ \nabla_{ad} = \frac{1 + (4 + \beta)(1 - \beta)/\beta^2}{5/2 + 4(4 + \beta)(1 - \beta)/\beta^2} \]

But this is written this way because the complex expression:

\( (4 + \beta)(1 - \beta)/\beta^2 \)

tends to \( 4/\beta^2 = \) infinity when \( \beta \to 0 \) (full radiation), and tends to 0 as \( \beta \to 1 \) (full gas). This makes it easy to find the limits:

\[ \nabla_{ad} \to 1/4 \text{ as } \beta \to 0 \quad \nabla_{ad} \to 2/5 \text{ as } \beta \to 1 \]

Plot shown below:
(e) [10 pt] Using the expression

$$\nabla_{rad} = \frac{3 \kappa L P}{16 \pi a c G m T^4}$$

and noting that $P_{rad} = 1/3aT^4$, so

$$\nabla_{rad} = \frac{1}{16 \pi c G m} \frac{\kappa L P}{P_{rad}} = \frac{1}{16 \pi c G} \frac{\kappa}{1 - \beta} \frac{L}{m}$$

For a star fully supported by radiation pressure, $\beta = 0$. Putting in numerical values and scaling to the Sun:

$$\nabla_{rad} = 6.7 \times 10^{-6} \left( \frac{L}{M} \right)_\odot$$

where $(L/M)_\odot \approx 2 \text{ erg/s/g}$. setting this equal to 0.25 gives

$L/M \approx 3.7 \times 10^4 (L/M)_\odot = 7.4 \times 10^4 \text{ erg/s/g}$

i.e., much greater than the Sun. The Eddington limit is:

$$L_{edd} = \frac{4 \pi G M c}{\kappa}$$

Notice that this means we can just write the radiative del as:

$$\nabla_{rad} = \frac{1}{4} \frac{L}{L_{edd}} \frac{M}{m}$$

So the radiative del and adiabatic del for radiation are equal when we hit the Eddington limit. Beyond this, the star is moving material. This is OK in the interior, but if this happens at the surface, the star convects itself to pieces! The plot to the right compares $\text{del}_\text{ad}$ to $\text{del}_\text{rad}$ for different values of $L/M$. 
(4) MESA: The Envelopes of Massive Giant Stars [40 pt]

Example inlists are provided on the course website

(a) Mass dependency on envelope inflation: [15 pts]

<table>
<thead>
<tr>
<th>Mass ($M_\odot$)</th>
<th>Time from ZAMS to H depletion (Myr)</th>
<th>$R/R_{ZAMS}$</th>
<th>$L/L_{Edd}$</th>
<th>Model number at H depletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>19.5</td>
<td>2.1</td>
<td>0.38</td>
<td>253</td>
</tr>
<tr>
<td>20</td>
<td>7.5</td>
<td>2.5</td>
<td>0.77</td>
<td>271</td>
</tr>
<tr>
<td>40</td>
<td>4.3</td>
<td>3.8</td>
<td>0.94</td>
<td>284</td>
</tr>
<tr>
<td>60</td>
<td>3.3</td>
<td>5.2</td>
<td>0.96</td>
<td>302</td>
</tr>
<tr>
<td>80</td>
<td>2.9</td>
<td>7.7</td>
<td>0.96</td>
<td>326</td>
</tr>
</tbody>
</table>

10 $M_\odot$: [Diagram]
$20 \, M_\odot$:

$40 \, M_\odot$:
$60 \, M_\odot$: 

![Graph 1](image)

$80 \, M_\odot$: 

![Graph 2](image)
Some observed trends:

$L/L_{\text{Edd}}$ increases up to 40 $M_{\odot}$; beyond this, the luminosity is just below Eddington at the end of H-burning, but significantly above Eddington for much of the H-burning phase for 60 $M_{\odot}$ and 80 $M_{\odot}$.

$R/R_{\text{ZAMS}}$ increases steadily with mass across the whole range, with the largest radii at the end stages of H-burning. The increase in $R$ becomes sharper for higher mass stars.

(b) The role of mixing length on stellar structure: [15 pts]

Here is the results of these calculations

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Time from ZAMS to H depletion (Myr)</th>
<th>$R/R_{\text{ZAMS}}$</th>
<th>$L/L_{\text{Edd}}$</th>
<th>Model number at H depletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19.5</td>
<td>1.8</td>
<td>0.405</td>
<td>253</td>
</tr>
<tr>
<td>1.5</td>
<td>19.5</td>
<td>1.9</td>
<td>0.40</td>
<td>253</td>
</tr>
<tr>
<td>3</td>
<td>19.5</td>
<td>1.9</td>
<td>0.37</td>
<td>253</td>
</tr>
<tr>
<td>10</td>
<td>19.5</td>
<td>1.9</td>
<td>0.35</td>
<td>253</td>
</tr>
<tr>
<td>100</td>
<td>19.5</td>
<td>1.9</td>
<td>0.34</td>
<td>253</td>
</tr>
</tbody>
</table>

The only significant change we see is in the luminosity fraction, which is a real change in the luminosity itself. The higher the $\alpha$, the lower the luminosity. One possible reason for this is that the longer mixing length scale may more efficiently carry heat outward, reducing the overall temperature gradient and resulting in a cooler core and hence a reduced nuclear generation rate.

(c) Internal structure of high-mass star [10 pts]:

Here are the plots generated with profile_panels1 at:

ZAMS (initial model):
50% H-depletion (around model 87):

90% H-depletion (around model 200):
The structures for the 60 $M_\odot$ star case is shown above. Some key observations:

1. The density profile flattens from ZAMS to H-depletion; this is largely caused by the expansion of the outer layers of the star as fusion stars to move toward a shell format.
2. The peak opacity spreads outward and broadens, corresponding to cooler outer layers.
3. The relative $L/L_{\text{Edd}}$, close to one at the surface, becomes $\approx 1$ over a larger fraction of the star (outside $r_{\text{core}}$) as the opacity increases.
4. In these regions, pressure is increasingly supplied by radiation, not gas. The overall fraction of gas/radiation pressure declines throughout the star as it evolves and generates a higher overall luminosity as the core heats.