Outline for Lecture 10:

Announcements:

• Make-up lecture tomorrow
• HW3 due Friday

Review – energy generation rate based on Gamow peak approximation
Parameterizing energy generation rates
Electron screening
Resonance – nuclear shell model
Nuclear chains
Abundance evolution
We have been discussing the last of our state equations, the energy generation rate per unit mass or $E = E(p, T, X)$. We have seen that nuclear energy production is possible due to strong binding energies from strong force potential, resulting in a net energy gain at order 4-8 MeV/nucleon. Particles achieve this state by quantum tunneling through the ~1 MeV Coulomb barrier at $r \approx 10^{-13}$ cm as too few particles are energetic enough to be above the potential at typical stellar core temperatures ($\sim 10^7$ K). The tunneling probability scales as $e^{-(E_0/E)^{1/2}}$, and combined with the Maxwell-Boltzmann distribution $e^{-E/kT}$ term we find the distribution of particles is concentrated in the Gamow peak.

$E_0 = 1.22 \frac{Z}{A} \frac{Z}{A} \frac{2}{3} \frac{2}{3} M_\odot \frac{1}{3} \text{ keV}$

$\Delta = 2.3 (E_{0\text{MeV}})^{1/4}$

Reaction rates are then computed as:

$$\nu_{\alpha \beta} = \frac{N_{\alpha} n_x <\sigma v_\beta>}{1 + \delta_{\alpha \beta}}$$

Kroenkecher delta

$$\frac{\rho^2}{m_n^2} \frac{X_\alpha X_x}{A_\alpha A_x} \frac{<\sigma v_\beta>}{(1 + \delta_{\alpha \beta})}$$

mass fraction
The energy production rates are then

\[
E_{\text{ax}} = \frac{\dot{V}_{\text{ax}} \cdot G_{\text{ax}}}{\rho} = \frac{\rho}{m_{\text{a}}^2} \frac{A_{\text{a}} \cdot A_{\text{x}}}{A_{\text{a}} \cdot A_{\text{x}}} \cdot \frac{\exp(-E_{\text{x}})}{(1+\delta_{\text{ax}})}
\]

... with net energy produced per reaction

\[
= \Delta BE - E_{\nu}
\]

It is customary to parameterize \( E \) as:

\[
E = E_0 \rho^a T^b
\]

The dependence comes largely through \( \langle \sigma v \rangle \). If we approximate Gamow recoil as Gaussian, this yields

\[
\langle \sigma v \rangle = \frac{0.72 \times 10^{-16}}{A_{\text{a}} A_{\text{x}}} \frac{e^{-\frac{T}{T_0}}}{\frac{e^2}{c^2} \text{ cm}^2/\text{s}}
\]

\[
T = \frac{3kT}{e} \text{ keV}
\]

\[
\alpha = e^{-\frac{T}{T_{1/3}}}
\]

\[
\Rightarrow \frac{\dot{V}}{\alpha} = \frac{\ln(4)}{3 T_0^{1/3} - 2/3}
\]

For typical reactions we'll discuss, here's \( A, \nu \):

<table>
<thead>
<tr>
<th>Reaction</th>
<th>( A )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP chain</td>
<td>1</td>
<td>( \approx 4 ) for ( T \approx 1.5 \times 10^8 ) K</td>
</tr>
<tr>
<td>CNO chain</td>
<td>1</td>
<td>( \approx 10 ) for ( T \approx 2 \times 10^8 ) K</td>
</tr>
<tr>
<td>3\alpha cun</td>
<td>2</td>
<td>( \approx 40 ) for ( T \approx 10^8 ) K</td>
</tr>
</tbody>
</table>

Note: the \( \langle \sigma v \rangle \) is generally determined experimentally from laboratory experiments. However, to obtain sufficient rates, \( T > 10^8 \text{ K} = 10^9 \text{ K} \) is a major extrapolation down to stellar energies!
Resonant Reactions

At certain energies, reaction rates can spike due to a resonant, discrete state in the intermediate nucleus $E^*$:

$$\alpha + X \rightarrow N^* \rightarrow Y + \beta$$

\[ \uparrow \quad \text{discrete state } E^* \quad \text{resonant energy} \]

$$\Rightarrow M^2 \gamma^2 + E^* = M_X^2 \gamma^2 + M_Y^2 + E_r$$

($E^*$ might be a particular nuclear shell state, as we will discuss.) Then the shape function $S(E)$ can take on a different form:

$$S(E) = \frac{\Gamma}{S_0 (E-E_r)^2 + (S_0^2)^2}$$

where $\Gamma = \frac{k}{\tau_{\alpha}}$ is the energy width of the reaction, $\tau_{\alpha}$ is the mean lifetime for reaction, the inverse sum of all possible reaction lifetimes:

$$\frac{1}{\tau_{\alpha}} = \sum_{i} \frac{1}{\tau_i} = \Gamma = E \Gamma_i$$

Typically, $E_r$ is measured in MeV to GeV, $\Gamma$ in keV.

If cross-section strongly peaked at $E_r$, then integral for $\langle 0U \rangle$ becomes:

$$\langle 0U \rangle \approx \sqrt{\frac{5}{16 \pi}} \frac{\Gamma}{S_0} \int_0^{\infty} \frac{g(2E+1) e^{-E_r/\Gamma}}{(E-E_r)^2 + (S_0^2)^2} dE$$

$$\approx 2.56 \times 10^{-13} \left( \frac{q_{\alpha} \Gamma}{M_T Q} \right)^{1/2} E^{-1.605E_r/T_q} \text{ cm}^3/s$$

$$E = g(2E+1) \frac{\Gamma \alpha \beta}{\Gamma} \text{ in MeV}$$

tabulated for various reactions.
Notes:
1. Resonances typically occur for high $E$ in small $A$ reactants, lower $E$ for high $A$ reactants - depends on the nuclear structure of intermediate nucleus.
2. If a resonance is within energy range of particle, it typically dominates all reactions.
3. Decaying intermediate state typically have several "exit" channels - it is usually the channel with the shortest $T_{1/2}$ (and $T$) that dominates.

Electron screening

Another factor to consider in stars is the screening effect of the sea of (possibly partially degenerate) electrons, which reduces Coulomb potential:

$$ U = \frac{2 e^2}{r} e^{-k_d r} $$

Debye scale $k_d = \frac{4 \pi e^2}{\mu T}$

For typical stellar interiors, $\frac{4 \pi e^2}{\mu T} \approx 4 \times 10^{-17} \text{ cm}^{-2}$

$\Rightarrow k_d \approx 2 \times 10^{-9} \text{ cm} \approx 10^{-11} \text{ cm}$

$\Rightarrow U \approx \frac{7 \times 2 e^2}{r} \left( 1 - k_d r \right)$

$= U_0 - U_0 e^{-7 \times 2 e^2 k_d} \approx 100 \text{ eV}$

This changes to support for our energy distribution, so that:

$$ <\sigma v> = \int_{u_0}^{\infty} (E - U_0) \frac{1}{\sqrt{2 \pi \sigma}} \exp \left( - \frac{(E - U_0)^2}{2 \sigma^2} \right) \text{d}E \approx e^{U_0/\sigma} <\sigma v>_0 $$

$\Rightarrow$ enhancement of reaction rate by $e^{U_0/\sigma}$ x 10-20 x at higher
Nuclear Shell Model

We've discussed nuclear reactions generally, but it is important to assess what reactions are actually occurring inside stars, and why, since many reactions are in principle possible. For this, we need to look at the nuclear shell model in detail.

1963 Nobel Prize: Wigner, Mayer + Jensen

~ UMD faculty, one of only two women to receive Nobel Prize in Physics

- Basic concept: nuclear potential is a combination of:
  - blend of harmonic oscillator + infinite potential well
  - spin-orbit coupling

### Harmonic Oscillator

- Potential: \( V = V_0 + \frac{1}{2} M \omega^2 r^2 \)
- Energy Levels: \( E(n \ell, J) = V_0 + n \omega (2n \ell + \ell + 1) \)
- \( \ell + \frac{1}{2} \text{ MeV/}A^{1/3} \)

### Infinite Well

- Potential: \( V = \begin{cases} -V_0 & r < R \\ 0 & r > R \end{cases} \)
- Energy Levels: \( E(n \ell, J) = V_0 + \frac{k^2}{2M} \left[ \ell^2 (\ell + 1/2)^2 - \ell (\ell + 1) \right] \)
- \( k = \frac{2\pi}{\ell} \)
- \( J \) is angular momentum
- \( \ell \) is spin

### Spin-Orbit

- Potential: \( V_{SO} = -a \ell \cdot s \)
- \( a \text{ MeV/}A^{1/3} \)

In both cases, we end up with discrete energy states split further by spin-orbit coupling.
The combination of principle energy levels (square/harmonic) and spin-orbit separation (which are significant!) gives rise to gaps at irregular intervals, when total nucleon numbers are 2, 6, 10, 20, 22, 50, 82, 126 → these are the magic numbers and result in unusually stable nuclei due to large energy spacing.

e.g. \[ \frac{1}{2} \text{He} = 2p + 2n = (1s_{1/2})^2 \]
\[ ^{16}_8 \text{O} = 8p + 8n = (1s_{1/2})^2 (1p_{3/2})^4 (1p_{1/2})^2 \]
\[ ^{40}_{20} \text{Ca} = 20p + 20n = (1s_{1/2})^2 (1p_{3/2})^4 (1p_{1/2})^2 (1d_{5/2})^6 (2s_{1/2})^2 (1d_{3/2})^4 \]
\[ ^{50}_{20} \text{Ca} \quad \text{not stable}! \]
\[ ^{50}_{20} \text{Ni} = 28p + 22n \rightarrow \text{not stable}! \quad \text{but abundant in SN } 2a \]
\[ ^{208}_{82} \text{Pb} = 82p + 126n = \text{doubly magic (and stable)} \]
\[ ^{208}_{50} \text{Sn} = 10 \text{ stable isotope}! \]
The shell model allows us to determine the spin and parity of a nucleus, important for assessing decay channels. "Closed shells" have $J=0$, $\Pi = 1$ (even), and even numbers of protons and even number of neutrons have $J=0$.

**E.g.** $^{17}O$ in protons: \[(1s^2)^2(1p^2)^2(1s^2)^2 \Rightarrow J=0, \Pi = 1\] closed closed

In neutrons: \[(1s^2)^2(1p^3)^2(1s^2)^2(1s^2)^2 \Rightarrow J = \frac{5}{2}, \Pi = (-1)^{2.1} = 1\]

For a reaction that combines particle $A + X$ to form $Z$ via strong force, total angular momentum and parity are conserved, i.e.

$\vec{J}_Z = \vec{J}_A + \vec{J}_X + \vec{L}$ relative angular momentum between $A + X$

$\Pi (Z) = \Pi (A) \Pi (X)(-1)^L$

This can dictate the type of final + angular distribution of reactions:

1. **Other rules:**
   1. Strong force preserves $J \pm \Pi$
   2. Fermionic opposite parity, $\Delta J = 1$ (even amm)
   3. Weak force: flips parity (Yang & Lee 1957 Nobel Prize)
   4. Like particles: Fermions antisymmetric, boson symmetric

   $\Rightarrow \uparrow X + \downarrow X$ in $1S_0$ state

   $\Delta J = 0 \Rightarrow$ even parity

   $\Rightarrow$ spin of antisymmetric

   $\Rightarrow \frac{1}{\sqrt{2}} (\uparrow + \downarrow)$ only (singlet state)
Nuclear reaction chains

Nucleosynthesis + decay can proceed along several chains through step, mitigated by EM, strong + weak forces

Weak interactions are as they sound - weak, unlikely, so these tend to be rate-limiting steps. Also \( e^- / e^+ \) emission is accompanied by \( \bar{e} / e \) emission, and neutrino energy is generally lost from star (except in ultradense cores undergoing SN shock).

Most reactions are hybrid, in the + reactants \( \rightarrow \) products through and intermediate channel: e.g.

\[
^1H + ^1H \rightarrow ^2He^* \rightarrow ^1H + ^0e + \nu_e
\]

- proton capture (strong)
- beta decay (weak)
- rate limiting step
- for total rate
Proton-proton chain

The lowest energy chain is $^1H$ fusion due to lower Coulomb barrier.

The $pp$ chain proceeds in three phases, with net reaction:

$$^1H + ^1H \rightarrow ^2He + ^1He + 2\nu_e + Q_{ff}$$

$$Q_{ff} = 13.116 \left[ 1 + 1.412 \times 10^6 \left( \frac{1}{T_9} - 1 \right) e^{-5/T_{9}^{9/4}} \right] \text{MeV}$$

<table>
<thead>
<tr>
<th>Phase</th>
<th>$^1H$ + $^1H$</th>
<th>$^2He$ + $^1He$ + 2$\nu_e$ + $Q_{ff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP I</td>
<td>$^1H + ^1H \rightarrow ^2He + ^1He + 2\nu_e$</td>
<td>1.442 MeV, 0.26 MeV</td>
</tr>
<tr>
<td>PP II</td>
<td>$^3He + ^4He \rightarrow ^7Be + \gamma$</td>
<td>1.580 MeV</td>
</tr>
<tr>
<td>PP III</td>
<td>$^7Be + ^1H \rightarrow ^8B + \gamma$</td>
<td>0.135 MeV</td>
</tr>
</tbody>
</table>

Let's look at a few of these reactions in detail:
\[ \text{\textit{H}} + \text{iH} \rightarrow (\text{\textit{He}}^2) \rightarrow \text{\textit{H}} + \text{i}e + \text{He} \]

This is a proton capture followed by a weak beta decay, and occurs so rarely that it is generally the rate limiting step for pp excess (S is 10 orders of magnitude below all other rates). Note \( \text{\textit{He}}^2 \) is highly unstable and generally breaks right back up into \( p + p \), but if weak interaction occurs during that time, then stable deuterium can form. The binding energy is:

\[ M_{\text{He}^2} - 2M_p c^2 - M_e c^2 = 0.42 \text{ MeV} \]

However, positron will combine with \( e^- \) to liberate \( 2M_e c^2 = 1.022 \text{ MeV} \). The neutrino carries away its KE; the distribution of energies is such that \( \overline{\nu_e} < 0.26 \text{ MeV} \rightarrow 1.82 \text{ MeV} \) net.

The small S for this reaction translates into \( \sigma = 10^{-47} \text{ cm}^2 \) at 1 MeV, a tipped laboratory energy. This makes this reaction effectively impossible to measure; 1 Amp flux of 1 MeV \( p \) would take 10 yr to create a significant reaction!

In the Sun, \( T \approx -\frac{n_p}{\text{cm}^3 / \text{dt}} = \frac{n_p}{2r_{pp}} \)

\[ r_{pp} = \frac{1.15 \times 10^4}{T_6^{7/2}} \times \rho^{2/3} e^{-3.38/T_6^{1/3}} \text{ cm}^3 / \text{s} \]

\[ \rho \times 10^{16} \text{ cm}^{-3} \]

\[ T_{pp} \approx 10^{10} \text{ yr} \leftarrow \text{right number for age of MS stars} \]

Without weak interaction, stars would either not burn or burn up completely.
\[ ^2\text{H} + ^1\text{H} \rightarrow ^3\text{He}^* \rightarrow ^3\text{He} + \gamma \]

This is a proton capture followed by an EM decay to excite \(^3\text{He}\) molecules - both are "fast" reactions. Because the reduced mass of \(^2\text{H} + ^1\text{H}\) is greater than \(^1\text{H} + ^1\text{H}\) (but Coulomb potential the same) the Gamow peak is shifted to higher energy, but ratio can proceed rapidly below \(10^7\) K.

Deuterium burning can occur before \(^1\text{H}\) burning - in stars too cool to fuse \(^1\text{H}\) (burns first). However, typical abundance of \(^2\text{H}\) and \(^3\text{He}\) are much too low to play a significant role in stellar evolution. Similarly \(^7\text{Li} + ^7\text{Be}\) fusion can occur at lower temperatures / lower mass stars.

\[ ^8\text{B} \rightarrow ^8\text{Be} \rightarrow ^2\text{He} + ^6\text{Li} + \gamma \]

This is a rapid decay that liberates considerable energy (18 MeV) and energetic \(^\gamma\) (7.2 MeV average). These neutrinos are sufficiently energetic to interact with \(^{37}\text{Cl} \rightarrow ^{37}\text{Ar}\), the reaction used in the Davis experiment to detect the neutrino flux from the Sun (and thereby \(\nu\) mixing and the nonzero rest mass of \(\nu\)).

\[ ^7\text{Be} + ^0\text{e} \rightarrow ^7\text{Li} + \nu\text{e} \rightarrow ^7\text{Li} + \gamma \]

The electron captured by \(^7\text{Be}\) is usually a free \(e^-\) - but can in principle be obtained from an inner bound \(e^-\). Since this is a reaction between positive - negative charge, this is strongly favored, which is why \(\text{PP} \, \text{II}\) is favored over \(\text{PP} \, \text{III}\) except in hottest stars.
Reactions + Abundances

Using these various chains, it is possible to infer the time-dependent abundances of elements at various regions in a star, keeping in mind that convection mixing may even these abundances out.

Let's consider the P-P chain alone:

\[ ^1\text{H} + ^1\text{H} \rightarrow ^2\text{H} + ^0\text{e} + ^1\text{e} \quad \text{rate} \quad r_{pp} = A_{pp} \frac{[\text{H}]^2}{2} \]

\[ ^2\text{H} + ^1\text{H} \rightarrow ^3\text{He} + ^0\text{e} \quad \text{rate} \quad r_{pd} = A_{pd} [\text{H}][\text{D}] \]

\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2^1\text{H} \quad \text{rate} \quad r_{33} = A_{33} \frac{[\text{He}]^2}{2} \]

Here \([x]\) is the number density of species \(x\) and \(dA/dt\) is just \langle 0\nu \rangle \langle x \rangle, a function primarily of temperature.

If this is a closed cycle then the number densities vary by:

1. \[ \frac{d[\text{H}]}{dt} = -2A_{pp} \frac{[\text{H}]^2}{2} - A_{pd} [\text{H}][\text{D}] + 2A_{33} \frac{[\text{He}]^2}{2} \]

2. \[ \frac{d[\text{D}]}{dt} = A_{pp} \frac{[\text{H}]^2}{2} - A_{pd} [\text{H}][\text{D}] \]

3. \[ \frac{d[\text{He}]}{dt} = A_{pd} [\text{H}][\text{D}] - 2A_{33} \frac{[\text{He}]^2}{2} \]

4. \[ \frac{d[^4\text{He}]}{dt} = A_{33} \frac{[\text{He}]^2}{2} \]

These are four simultaneous equations and four unknowns (assuming \(dA/dt\) are calculated), and assuming one starts with initial abundances you can in principle determine abundances over time.
In addition, for "carrier" elements, one can determine steady state abundances; take [D] for instance:

\[
\frac{d[D]}{dt} = 0 = A_{dp} \frac{[H]^2}{2} - A_{pd} [H][D]
\]

\[
\Rightarrow \frac{[D]}{[H]} = \frac{A_{dp}}{2A_{pd}}
\]

Recall that by approximating the non-resonant reaction rates as Gaussian over the Gamow peak, we get:

\[
\chi = <\sigma v> = \frac{7.2 \times 10^{-19}}{A_{pd}^2} \frac{S_0}{A_{pd}} e^{-T^2} \approx 42.4 \text{ cm}^3/\text{s}
\]

\[
T = \frac{350}{A_{pd}^2}
\]

\[
\Rightarrow \chi = 42.4 \frac{T_0^{-1/3}}{\left(A_{pd}^2 - A_{pp}^2\right)^{1/3}}
\]

For Sun, $T_0 = 15$, $A_{pp} = \frac{1}{2}$, $A_{pd} = 2/3$, $S_0 = 3.8 \times 10^{-22}$ km/s

\[
S_{pd} = 2.5 \times 10^{-46}
\]

\[
\Rightarrow \frac{[D]}{[H]} \approx 3 \times 10^{-18}
\]

More complex chains of reactions can be computed; see the famous

$\text{B}^3\text{He} - \text{B}^3\text{H} - \text{B}^3\text{Li} - \text{B}^3\text{Be} - \text{B}^7\text{Li}$

Note that if $[D] = [^3\text{He}]$ are in equilibrium, then

\[
\frac{d[^3\text{He}]}{dt} = \frac{1}{4} A_{pp} [H]^2
\]

i.e., production rate depends solely on rate to fuse H atoms