Outline for Lecture 13:

Announcements:
  • HW3 due tomorrow
  • Wednesday next week – workshop on literature research
  • Thursday next week: opacity paper (I-da)

Review luminosity relationship & energy generation rate
Energy transfer: radiation & random walk
Diffusion processes
Conduction
Convection
Set up & Thermo quantities
General “del” formalism
We now examine the temperature + radiative luminosity structure of a star, which of course allows us to understand the surface temperature & luminosity, important observable properties.

Consider a shell of gas again, and assume there is some process creating energy (gravity, nuclear, chemical, etc.). Define an energy generation rate:

\[ E = \frac{\text{energy generated}}{\text{time} \cdot \text{mass}} \quad \text{(erg/s/g)} \]

Power differential through slab:

\[ l(r+dr) - l(r) = \frac{\text{energy}}{\text{mass}} \]

\[ = 4\pi r^2 \rho E \]

or in mass units:

\[ \frac{dl}{dm} = E \]

We shall see that nuclear energy production can be well-described through power-law relations:

\[ E = 6\times10^{-6} T^4 \]

where (1,0) \( \rightarrow \) (1,4) \( \text{H} \rightarrow \text{He} \)

(1,15) \( \text{H} \rightarrow \text{He via CNO} \)

(2,40) \( \text{He} \rightarrow \text{C} \)
\[
\frac{dl}{dr} = 4\pi r^2 \sigma_0 \rho^{d+1} T
\]

This describes the energy production as a function of radius, but not how the energy is transferred.

**Eqn 4** Energy transfer or heat transfer

Transfer occurs through three methods:

1. **Radiation**: energy flux in an opaque medium occurs as a random walk of photons

   ![Diagram of radiation](image)

   The path length for the photon motion is defined as

   \[ l_0 = \frac{1}{K \rho} \]

   opacity \hspace{2cm} density
   \[ (\text{cm}^2/\text{g}) \hspace{2cm} (\text{g/cm}^3) \]

   In a convolved interior of a star, \( K \) due to Thomson scattering of free electrons:

   \[ K = \sigma_T n_e \]

   mass fraction of H

   \[ \sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{m_e c^2} \right)^2 \]

   \[ = 0.4 \text{ cm}^2/\text{g} \]

   \[ \Rightarrow l \approx 2 \text{ cm} \]
The flow of particles (photons, particle molecules) by random motion through a large path length is described by a diffusion equation:

\[ \vec{J} = -D \nabla n \]

\( J \) - flux of particles
\( n \) - number density
\( \nabla \) - gradient operator
\( D \) - diffusion constant
\( \langle \text{average velocity} \rangle \)

\[ D = \frac{1}{3} v l_p \leq \text{path length} \]

For radiation, treat this as an energy diffusion; then

\[ n \rightarrow U = \text{radiant energy density} \]

\[ = \alpha T^4 \quad \text{for blackbody radiation} \]

\[ v \rightarrow c \]

\[ l_p \rightarrow \frac{1}{4} \rho \]

\[ \Rightarrow F = -\frac{4 \pi c^3}{3 \hbar} \frac{dT}{dr} = \text{energy flux} \]

but \[ F = \frac{L}{4 \pi r^2} \implies \]

\[ \frac{dT}{dr} = -\frac{3}{16 \pi ac} \frac{k \rho l}{r^2 T^3} \]

Again, we will be able to write the opacity generally as

\[ \kappa = \kappa_0 \rho^{\eta} T^{-5} \].

\[ \Rightarrow \frac{dT}{dr} = -\frac{3 \kappa_0}{16 \pi ac} \rho^{\eta+1} T^{-(\eta + 3)} \]
Conduction can be treated similarly to radiation, as a diffusion equation:

\[ F_{\text{cond}} = -k_{\text{cond}} J \]

\[ = -\frac{4\pi e^2 T^3}{3 k_{\text{cond}} \rho} \]

Conduction generally only important for very high densities associated with degenerate matter:

\[ k_{\text{cond}} = 4 \times 10^{-8} \frac{\mu e^2}{\mu} \left( \frac{T}{\rho} \right)^2 \text{cm}^2/\text{s} \ll k_T \]

\[ \left( \text{on charge} \right) \]

Convection

Heat can also be carried by the gas if it circulates faster than the gas cooling time. As we will see later, this yields a \( T/P \) relation defined by the adiabatic gradient:

\[ \nabla a = \left( \frac{\partial \ln T}{\partial \ln P} \right)_{\text{ad}} = \frac{2}{5} \text{ for ideal gas} \]

In this case of (perfect) convection one obtains a polytropic relation \( P \propto \rho T \propto \rho P^{2/5} \)

\[ \Rightarrow P \propto \rho^{3/5} \Rightarrow n = \frac{3}{2} \]

More on this later...

Some comments:

1. Without energy generation, \( \frac{dL}{dr} = 0 \Rightarrow L \text{ constant; hence luminosity flatter out in ambient regions. Also } \frac{dL}{dr} \rightarrow 0 \text{ as } r \rightarrow 0 \)
(2) However, there is still a temperature gradient in a region of constant luminosity. Moreover, \( \frac{dT}{dr} < 0 \Rightarrow \text{temperature always hotter toward center.} \)

There will be cases where we may have an isothermal core (e.g. in the core in cooling MS star) where \( L = 0 \), but we'll never have an isothermal envelope.

(3) \( \text{Radiation + Convection can always be combined through a total opacity:} \)

\[
F_{\text{tot}} = F_{\text{rad}} + F_{\text{conv}} = \frac{4\pi ac}{3} \frac{T^3}{P} \left( \frac{1}{K_{\text{rad}}} + \frac{1}{K_{\text{conv}}} \right)
\]

\[
\Rightarrow \frac{dT}{dr} = -\frac{3}{16\pi ac} \frac{P}{T^3} \frac{K_{\text{eff}}}{K_{\text{conv}}} \leq \frac{1}{K_{\text{eff}}} - \frac{1}{K_{\text{conv}}} < K_{\text{conv}} + K_{\text{rad}}
\]

(4) \( \text{Radiation + Convection can be combined in gradient relation} \)

\[
\text{Grad} = \frac{dT}{dP} = \frac{dT}{dP} \frac{P}{dP} \frac{1}{T}
\]

\[
= \frac{-3}{16\pi ac} \frac{K_{\text{eff}}}{T^3} \frac{P}{P}
\]

\[
= \frac{GmP}{T^2} \frac{\frac{P}{T}}{P} - \frac{GmP}{T^2} \frac{T}{T}
\]

\[
= \frac{3}{16\pi ac} \frac{K_{\text{eff}}}{mT^4}
\]
Convection

Another way of transporting heat is to carry it with the gas. This can happen when the gas is sufficiently buoyant and rises faster than the timescale for heat to dissipate out. This simplified picture can be described by mixing length theory.

Some assumptions:

1. "Bubble" of gas has characteristic size \( l \) is scale of motion
2. \( l \) is small compared to, e.g., radius, \( \Delta p = \frac{\rho}{\partial r} \), etc.
3. Pressure inside and outside bubble remains the same
4. Temperature, density differ by small amount
5. Ignore external forces, shocks, etc., with exception of gravity (via buoyancy)
6. Parcels exchange no heat as it rises, depositing it all when it dissipates.

The picture is as follows:

\[
\frac{dT}{dT} = (\rho - \rho') Vg^2
\]

Assume a temperature gradient exists:

\[
\beta = \frac{dT}{d\rho} = -\frac{d\ln T}{d\ln p} = -\frac{\ln T}{d\ln p} \frac{dp}{dr} = -\frac{T}{dp}
\]

where \( \nabla = \frac{d\ln T}{d\ln p} \) and \( \Delta p = \frac{1}{\rho_0} \cdot \text{pressure scale height} \)

\[
\Delta p = \frac{1}{\rho_0}
\]
In a bubble, we can also describe a temperature gradient

$$\frac{dT'}{dr} = -T' \frac{dln T'}{dln p'} \frac{dln p'}{dr} = -T' \frac{J_{ad}}{dp}$$

where $J_{ad} = \frac{dln T'}{dln p'} \left( \frac{dln T}{dln p} \right)_{eq}$ is the adiabatic gradient.

Since we have assumed no heat exchange.

The adiabatic gradient is something that depends purely on the properties of the gas and some thermodynamics, and are computed as part of the general adiabatic expansion of a gas:

$$\Gamma_1 = \left( \frac{dln p}{dln p} \right)_{ad}$$

$$\frac{\Gamma_2 - 1}{\Gamma_2 - 1} = \left( \frac{dln p}{dln T} \right)_{ad} = \frac{1}{J_{ad}}$$

$$\Gamma_3 - 1 = \left( \frac{dln p}{dln p} \right)_{ad}$$

For a monatomic ideal gas, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3$

$$\Rightarrow J_{ad} = 0.4$$

Back to bubble: the bubble will continue to rise as long as the net buoyancy force continues to point upward, namely $p' < p$. This is the equivalent of requiring $\frac{dp'}{dr} < \frac{dp}{dr}$ and $\frac{dT'}{dr} > \frac{dT}{dr}$ for equivalent $P$

$$\Rightarrow T' \frac{J_{ad}}{dp} < T \frac{\partial}{dr}$$

$$\Rightarrow J_{ad} < \frac{\partial}{dr}$$

This is the Schwarzschild criterion (1906).
In regions where full convection applies:

\[ \frac{dT}{dr} = \frac{d}{dr} \left( \frac{dp}{md} \right) - \frac{T}{P} \left( \frac{\partial \ln T}{\partial \ln P} \right) \frac{G m_p}{r^2} \]

More generally:

\[ \frac{dT}{dr} = -\frac{T}{P} \sqrt{\frac{G m}{r^2}} \]

We can write a "cal" for radiation since:

\[ \frac{T}{P} \sqrt{\frac{G m}{r^2}} = \frac{3}{16 \pi a c G} \frac{k e p}{r^3} \]

\[ \Rightarrow \frac{T}{P} \sqrt{\frac{G m}{r^2}} \]

\[ \Rightarrow \frac{3}{16 \pi a c G} \frac{k e p}{m T^4} \]

And by analogy:

\[ \frac{3}{16 \pi a c G} \frac{k e p}{m T^4} \]

It is common in stellar models to compute \( \nabla, \nabla_{ad} \) as a function of radius:

Note heat flow follows minimum \( \nabla \) (i.e. \( \nabla = \min (\nabla_{ad}, \nabla_{conv}) \)).