Variable stars have been known since 16th century and encompass an entire genre of astronomical research.

Fabricius (1595) - “disappearing star” → Mira (“wonderful”)

Goodricke (1786) - 5 Cephei with 5 day period → Cepheid

Leavitt (1908) - Found relationship between period + L → “Standard Candle”

Today - American Association for Variable Star Observers (AAVSO)

discovery & plants through transit (Kepler)

Stars can vary due to magnetic spot, cloud, orbit, activity, etc.

One subset vary because of radial pulsation

Pulsation geometry - standing waves with node at center

Fundamental

1st overtone
What is the pulsation period?

\[ V_{\text{sound}} = \sqrt{\frac{8 \rho}{\rho}} \]

\[ R = \frac{c_i}{C_v} \]

Scaling:
\[ \frac{d \rho}{d r} \sim \frac{G M}{r^2} \Rightarrow \frac{\rho \dot{r}}{R} \approx \frac{GM \dot{r} \rho}{R^2} \times \frac{G \rho^2 R}{R} \]

\[ \Rightarrow \frac{\rho \dot{r}}{R} \approx \frac{G \rho^2 R}{R} \]

Period:
\[ \frac{R}{V_s} = \sqrt{\frac{\rho \dot{r}}{R}} = \sqrt{\frac{\rho \dot{r}}{R}} \approx \frac{R \dot{r}}{R \rho} \approx \frac{3 \pi}{2 R_0} \]

\[ \Rightarrow \text{period increase with lower density stars} \]

Where are pulsations seen?

\[ \text{inhabitable strips, predominantly affect giant and supergiant stars} \]

\[ \log L = 1.15 \log P + 2.47 \]

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Types:
- Long Period Variable (MLV)
  - \( P \approx 100-700 \) dy
- Cepheid
  - \( 1-500 \) dy
- RR Lyrae
  - \( 1.5-24 \) hr
- \( \beta \) Ceti
  - \( 100-3000 \) s
- white dwarf
Pulsation

We have largely considered our stars to be static objects, but there are many classes of variables of timescales of days to decades (and possibly longer).

One particular class of interest are regular pulsating stars (see chart).

To get a handle on this, consider a pure radial oscillation. The time scale can be inferred from our free fall or dynamical scale

$$t_{ff} \approx \sqrt{\frac{R}{g_0}} \approx 1 \text{ hour for Sun}$$

The density dependence! So dense stars have much shorter periods.

Consider only our main continuity and force equations,

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dp}{dr} = \frac{Gm_0}{r^2} - p_r$$

Consider only small departures from equilibrium ($\vec{v} = 0$) and perform a perturbation approximation:

$$r(t, M) \approx r_0(M) \left[ 1 + \frac{\delta r}{r_0} \right]$$

$$\rho(t, M) \approx \rho_0(M) \left[ 1 + \frac{\delta \rho}{\rho_0} \right]$$

$$P(t, M) \approx P_0(M) \left[ 1 + \frac{\delta P}{P_0} \right]$$

Mass equation:

$$\frac{dM}{d(r_0 \delta r_0)} = \frac{dM}{dr_0} \left[ 1 - \frac{dr}{dr_0} - r_0 \frac{d(r_0 \delta r_0)}{dr_0} \right]$$

$$\approx \frac{dM}{d(r_0 \delta r_0)} \approx \frac{dM}{dr_0} \left[ 1 + \delta r_0 \right] \left[ 1 + \frac{\delta P}{P_0} \right]$$

$$(\text{to order})$$
\[ \frac{d \eta}{d r_0} \left( \frac{1}{r_0} - \frac{dr_0}{ro} \frac{d (r_0/ro)}{d r_0} \right) = \frac{n r_0^2}{\pi} \left[ 1 + \frac{2 \frac{dr_0}{r_0}}{\frac{r_0}{ro}} + \frac{dr_0}{ro} \right] \]

Face eqn.:

\[
\frac{dp}{dr} = \frac{dp}{dr_0} \left( \frac{1 + \frac{dr_0}{r_0}}{\frac{dr_0}{r_0}} \right) = \left( \frac{dr_0}{r_0} \frac{d (r_0/ro)}{d r_0} \right) \frac{dp}{dr_0} \left( 1 + \frac{dr_0}{r_0} \right) \frac{d (r_0/ro)}{d r_0} + \frac{dp}{dr_0} \left( 1 + \frac{dr_0}{r_0} \right) \frac{d (r_0/ro)}{d r_0}
\]

\[
\frac{dp}{dr_0} = \frac{\sigma P_0}{r_0^2} \left[ 1 - \frac{dr_0}{r_0} \frac{d (r_0/ro)}{d r_0} \right] + \frac{\sigma P_0}{r_0^2} \left( 1 + \frac{dr_0}{r_0} - 2 \frac{dr_0}{ro} \right)
\]

\[
p_{r_0} \frac{d (r_0/ro)}{d r_0} = - \left( \frac{d \frac{dr_0}{r_0} + \frac{dr_0}{ro} }{\frac{dr_0}{r_0} + \frac{dr_0}{ro} } \right) \frac{d \frac{dr_0}{r_0}}{d r_0} - \frac{d (\frac{dr_0}{ro})}{d r_0}
\]

Assume Fourier components:

\[
\frac{dr}{ro} = S(r_0) e^{i \omega t} \quad \frac{dp}{ro} = S(r_0) e^{i \omega t} \quad \frac{dp}{ro} = S(r_0) e^{i \omega t}
\]

\[
\Rightarrow \frac{d^2 (r_0/ro)}{dt^2} = - \sigma^2 S(r_0) e^{i \omega t}
\]

We also need to tie \( p + P \) together by applying some condition,

\[
(\frac{\partial p}{\partial r_0})_{\partial_0} = \frac{\partial p}{ro} \Rightarrow \frac{\partial p}{ro} = \frac{\partial p}{ro} \frac{\partial p_0}{ro}
\]

Then the two:

1. \[
\frac{d \frac{dr_0}{ro}}{dr_0} = - \frac{1}{r_0} \left( \frac{d \frac{dr_0}{ro}}{dr_0} + \frac{1}{r_0} \frac{d \frac{dr_0}{ro}}{dr_0} \right)
\]

2. \[
\frac{d (\frac{dr_0}{ro})}{dr_0} \frac{d (\frac{dr_0}{ro})}{dr_0} = - \frac{\partial p_0}{ro} \left( \frac{d \frac{dr_0}{ro}}{dr_0} + \frac{d \frac{dr_0}{ro}}{dr_0} \right) \frac{\partial p_0}{ro} \frac{\partial p_0}{ro} \frac{\partial p_0}{ro}
\]

Pressure scale height
This is a pair of coupled 1st ODEs - what are boundary conditions?

\[ r = 0 \Rightarrow d \delta (r) = 0 \quad \text{(otherwise we obtain the star)} \]

\[ S = \text{constant} \]

\[ \text{from} \quad r = 0 \Rightarrow (3S + \frac{1}{\Gamma r} \xi \eta) \bigg|_{r=0} = 0 \]

\[ r = R \Rightarrow p \to 0 \]

but \[ \frac{dlp}{dr} = \frac{1}{\delta p} \to \infty \quad \text{as} \quad p \to 0 \]

\[ \Rightarrow (4S + \frac{\sigma^2 r^3}{\xi \eta} \delta \xi + \frac{\partial}{\partial r}) \bigg|_{r=R} = 0 \]

If we combine these two together we get a wave equation -

\[ L(\delta) = -\frac{1}{pr^4} \frac{d}{dr} (pr^4 \frac{d\delta}{dr}) - \frac{1}{r} \left[ \frac{d}{dr} (\delta \xi \eta \xi) \right] \delta = \sigma^2 \delta \]

This is the Linear Adiabatic Wave Equation and describes linear oscillation modes in an adiabatic gas constrained by the boundary condition above. Note that this equation includes the variable \( \delta (r) = \) spatial mode of wave, and \( \sigma^2 \), the eigenvalue of wave. We have to specify one of the other - the traditional approach is to assume

\[ \delta (r) = 1 \quad \text{for} \quad r = R \]

This is a forbidden equation, but we have some help in quantum mechanics theory because LWE is a Sturm-Liouville equation:

\[ \frac{d}{dx} \left[ p(x) \frac{dv}{dx} \right] + q(x) v = -M(x) v \]

Consequence of these expressions include:
1. Eigenvalues \( \lambda \), \( \lambda^2 \) are real \( \Rightarrow \) \( \sigma^2 > 0 \) \( \Rightarrow \) oscillatory
   or \( \sigma^2 < 0 \) \( \Rightarrow \) exponentially
   or \( \sigma^2 = 0 \) (no wave)

2. There is a minimum eigenvalue \( \lambda_1 \) and all other eigenvalues can be
   sorted from that \( \lambda_1 < \lambda_2 < \ldots < \infty \)

3. The eigenfunctions \( \gamma(x) \) are orthonormal:
   \[
   \int_a^b \gamma_n(x) \gamma_m(x) w(x) \, dx = \delta_{mn}
   \]
   \[
   = \int_0^1 \int_0^1 \gamma_n(x) \gamma_m(x) \, dx \, dm = 0 \quad n \neq m
   \]

\( S_n(r) \) therefore describe spherically symmetrical waves, with the # nodes set by \( \sigma^2 \)

**Example**

\[ \text{constant } S \neq R \]

\[ \Rightarrow \quad -\frac{1}{\rho} (3G - \gamma) \frac{df}{dr} s = \sigma^2 S \]

\[ \uparrow \frac{df}{dr} = \frac{6\sigma^2}{r^2} = -\frac{6\pi}{2} \int 6\rho^2 r \]

\[ \Rightarrow \quad (3G - \gamma) \frac{\text{const}}{3} \rho > \sigma^2 \]

\[ \Rightarrow \quad \text{period } P = \frac{2\pi}{\sigma} = \frac{2\pi}{\sqrt{(3G - \gamma) \rho > \sigma^2}} \approx \frac{1}{\sqrt{\rho \sigma}} \leq \text{free fall time!} \]

**Example:** polytropic \( P = \rho Y \Rightarrow \rho = \rho \frac{\gamma_n}{\gamma_n} \Rightarrow \rho \frac{\rho\gamma_n}{\gamma_n} \Rightarrow \rho = \gamma_n \frac{\gamma_n}{\gamma_n} \frac{\gamma_n}{\gamma_n} \]

Original equations (1):

\[
\frac{dG}{ds} = -\frac{1}{s} \left( 3s + \frac{1}{\rho} \frac{\partial s}{\partial \rho} \right)
\]

can be

\[
\frac{d(\sigma ons)}{ds} = -\frac{\sigma}{\rho} \frac{\partial \sigma}{\partial \rho} \frac{\gamma_n}{\gamma_n} \left[ 4s + \frac{\rho^2}{\gamma_n} \frac{\gamma_n}{\gamma_n} \frac{\gamma_n}{\gamma_n} \frac{\gamma_n}{\gamma_n} \right]
\]

\[ \omega^2 = \frac{\rho^3}{6M} \sigma^2 \]
Still a complex equation pair; can simplify LAWE by introducing ten
\[ w(r) = r^2 \sqrt{\frac{\pi}{1}} P S(r) \]

then LAWE becomes
\[
\frac{d^2 w}{dr^2} + \left[ \frac{\sigma^2}{\sqrt{\frac{\pi}{1}}} - \Phi(r) \right] w = 0
\]
\[
\frac{\pi}{1} \rho = \frac{\nu_s^2}{p}
\]
\[
dr(r) = \frac{\sigma}{r^2} + \frac{2}{\pi \rho r} \frac{d\rho(r)}{dr} - \left[ \frac{1}{2\pi \rho} \frac{d\rho(r)}{dr} \right]^2
\]
\[
+ \frac{1}{2\pi \rho} \frac{d\rho(r)}{dr} \frac{1}{\rho} \frac{d\rho(r)}{dr} \left[ (3\rho - u) \Phi(r) \right]
\]

let \[ u(r) \approx e^{i\omega r} \quad \text{(wave equation)} \]
\[ u(r) \approx k \quad \text{constant} \]

\[ \Rightarrow u^2 = \frac{\sigma^2}{\nu_s^2} - \Phi(r) \quad \text{dispersion relation} \]
\[ \sigma^2 > \nu_s^2 \Phi \Rightarrow \sinusoidal \]
\[ \sigma^2 < \nu_s^2 \Phi \Rightarrow \expansive (expansional) \]

Asymptotic relation \[ \sigma^2 \gg \nu_s^2 \Phi \Rightarrow kr \approx \sigma / \nu_s \]

Standing wave, \[ \int_a^b k r dr = \int_a^b \frac{\sigma}{\nu_s} dr \quad \text{integration sound speed through star} \]
\[ \Rightarrow \sigma = (n+1) \pi \left[ \int_a^b \frac{dr}{\nu_s} \right]^{-1} = (n+1) \frac{\sigma_0}{\nu_s} \]

Even spactial modes
\[ = \text{radial pressure modes} \]
Nonadiabatic Modes

The discussion prior explains how a star "rises" after it has been diminished, but what drives the oscillation? Here we have to examine non-adiabatic mode. These are generally driven by a heat engine.

\[
\begin{align*}
\text{driving} & \quad \text{for } P\text{d}v > 0 \\
\text{damping} & \quad \text{for } P\text{d}v < 0
\end{align*}
\]

To drive oscillation, it is necessary to add heat during high temperature portion of cycle (e.g., spark plug ignition during maximum compression).

2 ways to add heat:
- Internal energy - must be shell burning since center is a node - $C$ effect
- Opacity - if compression causes dropping $C$ heat - $K$ effect

Since energy transfers through an inviscid flow, go back to stellar shell energy:

\[
\frac{dL}{dm} = C_{\text{enc}} \cdot \Delta p
\]

\[
C = -\frac{P}{\rho (\gamma - 1)} \left[ \frac{\text{dln} P}{\text{d}t} - \frac{\text{dln} \rho}{\text{d}t} \right]
\]

\[
\rho (\gamma - 1) = \text{dln} P \Big|_{\text{d}t} \quad \rho (\gamma - 1) = \text{dln} \rho \Big|_{\text{d}p}
\]

For first term:

\[
\frac{P}{T} \text{ for } P \propto \rho \quad \frac{P}{T} \text{ for } P \propto \rho^{-1}
\]

\[
C_{\text{v}_p} = \frac{\text{dln} H}{\text{d}t} \text{ for ideal gas}
\]
then rewrite energy equation:

$$\frac{\Delta m}{\Delta t} = \frac{\Delta}{\Delta t} \left( P_1 \frac{\Delta}{\Delta t} \frac{\Delta}{\Delta t} \right) \left( e - \frac{\Delta}{\Delta t} \frac{\Delta}{\Delta t} \right)$$

or

$$\frac{\Delta m}{\Delta t} = (P_{\Delta} - P_0) \frac{\Delta}{\Delta t} \frac{\Delta}{\Delta t} + \frac{1}{C_v T} \left( e - \frac{\Delta}{\Delta t} \frac{\Delta}{\Delta t} \right)$$

linearize this: $T = T_0 + \Delta T$

and assume $\Delta^2$ small:

$P = P_0 + \Delta P$

$\varepsilon = \varepsilon_0 + \Delta \varepsilon$

$L = L_0 + \Delta L$

$$\Rightarrow \frac{\Delta (\varepsilon / T)}{\Delta T} = (P_{\Delta} - P_0) \frac{\Delta P}{\Delta T} + \frac{1}{C_v T} \left( \frac{\Delta \varepsilon}{\Delta T} \right)$$

eliminate term derivative by assume $\Delta (x) \approx e^\Delta x$

$$\Rightarrow \delta \varepsilon = -\frac{\Delta L}{\Delta T} = \omega C_v T \left[ \frac{\Delta T}{T} - (P_{\Delta} - P_0) \frac{\Delta P}{\Delta T} \right]$$

$$\Rightarrow \text{imaginary term}$$

$$\Rightarrow \omega \text{ must be complex & contain real & imaginary parts}$$

This factors into heat engine model; cherry $d$ need in cycle

$$\oint dG = \oint dE + \oint dw$$

Elliptic area one cycle

From $d$ done

If $T \rightarrow T_0 + \Delta T$ then

$$dS = \oint \frac{dE}{T} + \oint \frac{dG}{T_0} - \oint \frac{dT}{T_0} dG = 0$$

$$\Rightarrow dG = \oint \frac{dT}{T_0} dG$$

$$\Rightarrow W_{\text{net}} = \oint \frac{dT}{T_0} dG \quad \text{interpretation: work done}$$

$$\Rightarrow W_{\text{net}} = \int_{r_{\Delta t}} \oint \frac{dT}{T_0} dG dm$$

$$\Rightarrow \int_{r_{\Delta t}} \oint \frac{dT}{T_0} dG dm$$
What drives pulsation? - heat engine model

\[ \frac{\Delta P}{\Delta V} \]

\[ \begin{cases} \text{if } PdV > 0 \Rightarrow \text{driving} \\ \text{if } PdV < 0 \Rightarrow \text{damping} \end{cases} \]

Must add heat during high temperature portion of cycle (think spark plug ignition during maximum compression)

Could nuclear fusion provide heat? \( E \propto \rho^a T^b \approx \) greater for larger \( \rho \) & \( T \)

But center of star is a node - no compression or heating

Eddington heat value model: Opacity increases in compression, tapping heat in - possible in partial ionization zones

Recall \( \frac{\Delta N}{\Delta V} \propto \frac{1}{n_e} \frac{T^3}{e^{-\mu/\rho}} \)

Small increase in \( T \Rightarrow \) greater fraction \( \Delta V \)

\[ \Rightarrow \text{energy goes into ionization, not temperature} \]

\[ \Rightarrow \text{smaller } T \text{ increase} \]

If \( K \propto \rho / T^{3.5} \) (Kramers' rule)

\[ \frac{\Delta K}{K} \propto \frac{\Delta \rho}{\rho} \left( \frac{\Delta T}{T} \right)^{-3.5} \quad \text{&} \quad \frac{\Delta \rho}{\rho} > 0 \]
Partial ionization tone only occurs in stars over a narrow range of T.

Too deep and tone will have no influence on surface.

Too shallow and amount of mass to drive is too small.

\[ \Rightarrow \text{goldstone tone} \]
Let's exam heat transport equation

\[
\frac{dT}{dr} = -\frac{3}{16\pi Lc} \frac{kL}{r^2}
\]

Integrating as before we get

\[
\frac{\delta L}{L} = 4\delta S - \frac{\delta M}{M} + 4\frac{\delta T}{T} + \frac{1}{r} \frac{d}{dr} \frac{d}{dr} \frac{\delta L}{L}
\]

Assume \( k \propto r^{-s} \), \( \Gamma_{3-1} = \frac{\delta M}{M} \), \( \rho \), \( s \), \( -3s - \frac{\delta L}{L} \frac{d}{dr} \frac{d}{dr} \frac{\delta L}{L}
\]
\[
\Rightarrow \frac{\delta L}{L} = -\left( \frac{s+2}{\Gamma_{3-1}} \right) \frac{\delta T}{T} + (s+4) \frac{\delta T}{T}
\]

\[
= 4\frac{\delta T}{T} \quad \text{for known } \rho \text{ nonionized ideal gas}
\]

\[
\Rightarrow \frac{\delta M}{M} \sim 4L \frac{\delta (\delta T)}{\delta T}
\]

Here \( \delta T/T > 0 \), \( \delta L/L > 0 \Rightarrow \text{luminous, incandescent gas} \)

In central temperature regions, \( k \propto T^{-s} \) \( 1.5 < 7/4 \) and over regions

\[
\Gamma_{3-1} < 5/3
\]
\[
\Rightarrow \delta T/T > 0 \Rightarrow \delta L/L > 0 \Rightarrow \text{burn normal at core at } T
\]
\[\Rightarrow \text{heat engine}\]