Outline for Lecture 2:

Announcements:
- Homework #1 posted on website, due Friday 10/7 – there is use of data
- Schedule this week – no workshop, Thursday -> Friday @ 2:30pm
- Sign up for paper reports – Thursdays, rubric?

Building a model of a star
- Requires consideration of thermodynamics, fluid dynamics, radiative transfer, nuclear & particle physics, magnetohydrodynamics (if fancy), general relativity (if remnants)
- Assumptions:
  - Hydrostatic equilibrium (but we’ll explore)
  - Thermal equilibrium – energy loss \( \approx \) energy produced from fusion
  - Chemical near-equilibrium – time scale for abundance changes << time scale for star to adjust
  - Sphericity and angular isotropy (=> 1D) -> primary coordinate = \( r \)

Mass continuity
- Derivation
- Include time dependence -> spherically symmetric fluid equations
- Mass loss to fusion? Estimate rate (10% of H in 10 Gyr, 0.7% of mass lost => 1 part in 5e20 per second)
- Basic profile
- Lagrangian formalism = replace \( dr \) with \( dm \)

Force balance = hydrostatic equation
- Derivation
- Include time dependence, possible oscillation
- Time scales
- Piston model?

Energy conservation
- Define energy generation rate
- Nuclear => basic \( L(r) \) profile
- Heat loss/gain (e.g., change of state)
- Neutrino loss rate
- Time scales
- Virial theorem?

Energy transport
- Modes: radiation, conduction, convection, net transport of material
- General form of equation
- define delta
- derive delta for radiation?

Chemical equilibrium – nuclear rates
- Equations, define rates and link to energy generation rates
- Vertical distribution: mixing plays a role
State equations:
  • Equation of state – links density to pressure
    o ideal gas – \( \rho = \mu m/k \ P/T \)
    o radiation – \( P = aT^4 \)
    o polytrope – \( P = K \rho^{\gamma} = K \rho^{((n+1)/n)} \)
    o nonrel fully degenerate gas – \( P = 2/3 \ n \ E_{\text{fermi}} \sim \rho^{5/3} \)
  • thermal properties
    o adiabatic index \((P,T,X)\)
    o heat capacities \((P,T,X)\) - define
  • nuclear properties
    o rate equations \((P,T,X)\)
    o neutrino loss rate \((P,T,X)\)
  • radiative/ conductive properties
    o opacity kappa \((P,T,X)\)
    o conductive opacity \((P,T,X)\)
  • Variables to solve for: \( m(r), P(r), T(r), L(r) \)
    o Continuity for all of these
    o In addition: time dependence on \( X's \)

Boundary conditions:
  • \( m(0) = 0, m(R) = M \)
  • \( P(R) = 0? \rightarrow P_{\text{eff}} = 2/3 \ GM/R^2 \ 1/<\text{kappa}> \)
  • \( T(R) = T_{\text{eff}} \)
  • \( L(0) = 0, L(R) = L \)
Mass-Luminosity Relation for Main Sequence Stars

Curve shows $L = M^p$ with $p$ between 3 and 4

Adapted from data compiled by Svechnikov & Bessonova (1984)

Boyajian et al. (2012)
Consider a shell within which the mass contained is:

\[ dm = 4\pi r^2 \, dr \, \rho \]

\[ \Rightarrow \frac{dm}{dr} = 4\pi r^2 \, \rho \] (static)

If gas is moving, we must take into account net flow:

\[ dm = -4\pi r^2 \rho \, dr \, \frac{dt}{dt} = -4\pi r^2 \rho \, v \, dt \]

\[ \Rightarrow \frac{dm}{dt} = -4\pi r^2 \rho \, v \] (net change in mass, at constant radius)

Note that

\[ \frac{dm}{dr} = 4\pi \frac{d\rho}{dt} \, r^2 \]

\[ -8\pi r^2 \rho \, v = -4\pi r^2 \, \frac{d\rho}{dt} \, v - u \pi r^2 \frac{du}{dr} \]

\[ \Rightarrow \frac{d\rho}{dt} = -2 \rho \frac{v}{r} - \frac{d\rho}{dr} \, v - \rho \frac{du}{dr} \]

\[ = -\frac{1}{r^2} \frac{d}{dr} \left( \rho \frac{v^2}{r} \right) \]

\[ = -\frac{d}{dr} (\rho v) \] in spherical symmetry

The first equation also acts as an operator for switching between a **Eulerian description** (quantities as a function of space and time) and a **Lagrangian description** (quantities as a function of mass).
Equation 2: Hydrostatic Equilibrium

Consider forces acting on a slab arising from pressure + gravitational terms:

\[
\frac{dm}{dr} \cdot \frac{4\pi r^2 \rho}{dr} = -4\pi r^2 \left( P(r+dr) - P(r) \right) - (4\pi rt^2) g
\]

\[\Rightarrow \quad \rho \frac{d^2}{dr^2} - \frac{\partial}{\partial r} - \rho g \]

where \( \frac{G}{r^2} = \frac{G}{r^2} \int_0^r dr' \pi r'^2 \rho(r') \) (continuously equation)

This is in fact just the equation of fluid hydrodynamics:

\[
\rho \frac{d\vec{v}}{dt} = -\nabla P - \rho \frac{d\Phi}{dr}
\]

with \( \nabla \Phi = 4\pi G \rho \) (\( \Phi \) is the gravitational potential)

This equation could be applied for cases of non-spherical symmetry.

For hydrostatic equilibrium, \( \vec{r} = 0 \):

\[\Rightarrow \quad \frac{dp}{dr} = -\rho g - \frac{G \rho m(r)}{r^2}\]
Note that this automatically requires a pressure gradient that decreases outward!

Does hydrostatic equilibrium strictly apply?

Assume no pressure term: then \( |r'| = \frac{R}{\rho_{\text{eff}}} \approx g = G\rho R \) (collapse)

\[ T_{\text{ff}} \approx (G\rho)^{-\frac{1}{2}} \]

This is the free-fall time

Assume no gravity term: then \( \nu = \frac{R}{\nu_{\text{exp}}} \approx \frac{P}{\rho R} \) (expansion)

\[ T_{\text{exp}} \approx R \left( \frac{P}{\rho} \right)^{\frac{1}{2}} \]

\( R \ll C_s \) — speed of sound

\[ T_{\text{exp}} = T_{\text{ff}} \approx T_{\text{hyd}} \] (the star is in hydrostatic equilibrium)

For Sun \( T_{\text{ff}} \approx (G\rho_{\text{Sun}})^{-\frac{1}{2}} \)

\[ = \left[ (6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}) (1.4 \times 10^3 \text{ g/cm}^3) \right]^{-\frac{1}{2}} \]

\[ \approx 3000 \text{ s} < 1 \text{ hr} \]

\[ T_{\text{exp}} \approx \left( 7 \times 10^{15} \text{ cm} \right) \left( \frac{1.4 \times 10^3 \text{ g/cm}^3}{10^{16} \text{ dyn/cm}^2} \right)^{\frac{1}{2}} \]

\[ \approx 800 \text{ s} \]

These are comparable, and more importantly very fast, so any departure from equilibrium is very quickly corrected for either through free-fall or outward expansion.
Energy Generation + Transport

We now examine the temperature + radiative luminosity structure at a star, which of course allows us to understand the surface temperature & luminosity, important observable properties.

\[ E = \frac{\text{energy generated}}{\text{time} \cdot \text{mass}} \quad \text{(ergs/1g)} \]

Consider a shell of gas again, and assume there is some process creating energy (gravity, nuclear, chemical, etc.). Define an energy generation rate:

\[ E = \frac{\text{energy generated}}{\text{time} \cdot \text{mass}} \quad \text{(ergs/1g)} \]

Power differential through slab:

\[ L(r+dr) - L(r) = Edm \]

\[ = \int 4\pi r^2 P(r)dr \]

\[ \Rightarrow \frac{dL}{dr} = 4\pi r^2 P \]

or in mass units:

\[ \frac{dL}{dm} = E \]

We shall see that nuclear energy production can be well-described through power-law relations:

\[ E = 60\rho^{1/2}T^3 \]

where

1. \((d,p)(1,4) \text{H} \rightarrow \text{He} \)
2. \((1,15) \text{H} \rightarrow \text{He} \) via CNO
3. \((2,40) \text{He} \rightarrow \text{C} \)
\[
\frac{dL}{dr} = 4\pi r^2 L_0 \rho r^{d+1}
\]

This describes the energy production as a function of radius, but not how the energy is transferred.

Eqn 4

Energy transfer or heat transfer

Transfer occurs through three methods:

1. Radiation: energy flux in an opaque medium occurs as a random walk of photons.

The path length for the photon motion is defined as

\[
l_0 = \frac{1}{\kappa \rho}
\]

opacity \quad density
\[
(\text{cm}^2/\text{g}) \quad (\text{g}/\text{cm}^3)
\]

in ionized interior of Sun, \( \kappa \) due to Thomson scattering at free electrons:

\[
\sigma = \frac{8\pi}{3} \left( \frac{e^2}{m_c c^2} \right)^2
\]

\[
\sigma \approx 4 \times 10^{-17} \text{ cm}^2/\text{g}
\]

\[
\Rightarrow l = 2 \text{ cm}
\]
(3) Convection

Conduction can be treated similarly to conduction as a diffusion equation.

\[ F_{\text{cond}} = -k_{\text{cond}} \nabla T \]

\[ = -\frac{4\pi e T^3}{3 \ k_{\text{cond}} \rho} \]

Conduction generally only important for very high densities associated with degenerate matter:

\[ k_{\text{cond}} = 4 \times 10^{-8} \frac{\mu e^2}{\lambda} \left( \frac{T}{\rho} \right)^3 \left( \frac{\epsilon}{\mu^2} \right) \ll K_T \]

\[ T \text{ (on charge)} \]

Heat can also be carried by the gas if it circulates faster than the gas cooling time. As we will see later, this yields a \( T/P \) relation defined by the adiabatic gradient:

\[ \frac{\partial \nu}{\partial \mu} = \left( \frac{\partial \ln T}{\partial \ln P} \right)_{\text{ds}=0} = \frac{2}{5} \text{ for ideal gas} \]

In this case of (perfect) convection, one obtains a polytropic relation \( P \propto \rho T \propto \rho P^{2/5} \).

\[ \Rightarrow P \propto \rho^{3/5} \Rightarrow n = 3/2 \]

More on this later...

Some comments:

(3) Without energy generation, \( \frac{dL}{dr} = 0 \Rightarrow L \text{ constant, hence luminosity flattens out in ambient region. Also } \frac{dL}{dr} = 0 \text{ at } r = 0 \)

\[ \frac{dL}{dr} \rightarrow k \Rightarrow L \sim r^k \]
(2) However, there is still a temperature gradient in a region of constant luminosity. Moreover, \( \frac{dT}{dr} < 0 \Rightarrow \) temperature always hotter toward center.

There will be cases where we may have an isothermal core (e.g., late in the core in cooling MS star) where \( L = 0 \), but we'll never have an isothermal envelope.

(3) Radiation + Conduction can always be combined through a total opacity:

\[
F_{\text{total}} = F_{\text{conv}} + F_{\text{rad}} = \frac{4\pi acT^3}{3} \left( \frac{1}{K_{\text{conv}}} + \frac{1}{K_{\text{rad}}} \right)
\]

\[
\Rightarrow \frac{dT}{dr} = -\frac{3}{16\pi ac} \frac{\beta}{T^3} \frac{\rho}{M} \text{Kels} \leq \frac{1}{K_{\text{eff}}} = \frac{1}{K_{\text{eff}}} \leq \frac{1}{K_{\text{conv}}} + \frac{1}{K_{\text{rad}}}
\]

(4) Radiation + Convection can be combined in gradient relation:

\[
\text{Grad} = \frac{dT}{d\ln r} = \frac{\frac{dT}{dr} P}{\frac{dP}{dr}} \frac{\rho}{T}
\]

\[
= -\frac{3}{16\pi ac} \frac{\beta \rho L}{T^3} P
\]

\[
= \frac{GMr}{P} \frac{\rho}{T}
\]

\[
= \frac{3}{16\pi ac\gamma} \frac{\rho L P}{m T^4}
\]
The flow of particles (photons, particle molecules) by random motion through a large path length is described by a diffusion equation:

\[ \vec{J} = -D \nabla n \]

flux \( \vec{J} \)

diffusion constant

\[ \nabla \text{number density} \]

\[ D = \frac{1}{3} V l_p \leq \text{path length} \]

for radiation, treat this as an energy diffusion; then

\[ n \rightarrow U = \text{radiant energy density} \]

\[ \sim aT^4 \quad \text{for blackbody radiation} \]

\[ v \rightarrow c \]

\[ l_p \rightarrow \frac{1}{k_B} \]

\[ \Rightarrow F = -\frac{4acT^3}{3k_B} \frac{dT}{dr} = \text{energy flux} \]

but \[ F = \frac{L}{4\pi r^2} \quad \Rightarrow \]

\[ \frac{dT}{dr} = -\frac{3}{16\pi ac} \frac{k_B l_p}{r^2T^3} \]

\[ \text{again, we will be able to write the opacity generally as} \]

\[ \kappa \equiv K_0 \rho^{n+1} T^{-5} \]

\[ \Rightarrow \quad \frac{dT}{dr} = -\frac{3K_0}{16\pi ac} \rho^{n+1} T^{-(n+3)} l \]
Hydrostatic Eq. and the Virial Theorem

The equation for hydrostatic equilibrium is actually a starting point to derive the virial theorem, which we used to derive the luminosity time scale for gravitational contraction (the Kelvin-Helmholtz timescale).

Multiply both sides of Eqn 2 by \( \frac{r}{p} \) and integrate w.r.t. mass,

\[
\int_0^M \frac{dp}{dr} \frac{r}{p} \, dm = \int_0^M \frac{dp}{dm} \frac{dr}{p} \, dm
\]

\[
= \int_0^M dp \ [4\pi r^2 \rho] - \int_0^M p \ 12\pi r^2 \frac{dr}{dm} \, dm
\]

\[
= 4\pi R^3 \rho_s - 3 \int_0^M \frac{p}{p} \, dm
\]

pressure at surface

Consider an ideal gas: \( P = n k T \frac{f}{\mu} k T \)

\[
\Rightarrow \frac{P}{\rho} = \frac{k T}{\mu}
\]

mass element \( dm = \mu \, dN \) counting particles

and average kinetic energy per particle \( \bar{KE} = \frac{3}{2} k T \)

\[
\Rightarrow 3 \int_0^M \frac{p}{\rho} \, dm = 2 \int_0^{N_{\text{tot}}} \frac{\bar{KE}}{\mu} \, dN = 2 \bar{KE}_t
\]

Total kinetic energy
Right side: \[ - \int_0^M \frac{G \rho m(r)}{r^2} \, dm = - \int_0^M \frac{G m}{r} \, dm \]

Gravitational P.E. at radius r per mass element

\[ = \Omega \cdot \Omega \cdot \text{total gravitational potential energy} \]

\[ = 4\pi R^3 \rho_S = 2KE_a + \Omega \cdot \Omega \]

Visual theorem

Note that typically \( \rho_S = 0 \) (Space) \( \Rightarrow \) \( 2KE_a = -\Omega \).

Total energy \( E = KE_a + \Omega \), \( E = \frac{1}{2} \Omega \).

\( \Rightarrow \) if \( \frac{dE}{dt} > 0 \) (contraction), \( \frac{dE}{dt} < 0 \) = radiation

\( \frac{1}{2} \) of gravitational potential released goes to radiation

\( \frac{1}{2} \) heating interior

this happens during star formation phase.