Polytrope Models

An important form of the Eddington & Saha which makes stellar structure modeling amenable to analytic solutions is the polytrope, where pressure and density are related as:

\[ P \propto \rho^n \]

This appears in many situations:

1. Giant molecular cloud: As we've discussed a GMC is a low density, isothermal molecular gas, well described by Maxwell-Boltzmann statistics (i.e. ideal gas):

\[ P = \frac{\rho kT}{\mu m_H} \propto \rho \quad \text{(since } T \text{ and } \mu \text{ are constant)} \]

2. Star supported by gas + radiation pressure: Using the Planck distribution (Bose-Einstein distribution for photons) you can model derive that

\[ P_{\text{rad}} = \frac{1}{3} \alpha T^4 \]

\[ \Rightarrow P_{\text{gas}} + P_{\text{rad}} = \frac{\rho kT}{\mu m_H} + \frac{1}{3} \alpha T^4 \quad \text{"Eddington Standard Model"} \]

Define \( \beta = \frac{P_{\text{gas}}}{P_{\text{rad}}} \), \( 1 - \beta = \frac{P_{\text{rad}}}{P_{\text{gas}}} \)

\[ \Rightarrow T = \beta P \frac{\mu m_H}{\rho n} \]

\[ \Rightarrow \rho = \frac{P_{\text{rad}}}{1 - \beta} = \frac{1}{3} \alpha (1 - \beta) \left( \frac{\rho kT}{\mu m_H} \right)^{\gamma} \]

\[ \Rightarrow P^2 = \frac{3}{\alpha} (1 - \beta) \left( \frac{\rho kT}{\mu m_H} \right)^{\gamma} \rho^{\gamma/2} \]

\[ \Rightarrow P = \left( \frac{3}{\alpha} \right)^{1/2} (1 - \beta)^{1/2} \left( \frac{\mu k}{\rho \mu m_H} \right)^{\gamma/2} \rho^{\gamma/2} \propto \rho^{\gamma/2} \text{ if } \beta \text{ is constant} \]

For Sun \( \beta = 1 \) (\( P_{\text{gas}} \gg P_{\text{rad}} \))
Degenerate gas: Characteristics of brown dwarfs, giant planets, white dwarfs

\[ \frac{n_0}{n_F} \]

\[ e_F \quad e \]

In HW you will derive \( E_F \sim (3\pi^2 n_i^2)^{2/3} \frac{k^2}{2m} \)

\[ P \quad \frac{1}{3} \int_{e_F}^{n_F} \frac{n}{e_F} \frac{dE}{d\mathbf{v}} \frac{dE}{2E} \text{ (for classical gas)} \]

\[ \frac{2}{3} n_E^2 \]

\[ \frac{1}{3} \left( \frac{L_{mm}}{\mu_{nm}} \right)^{2/3} \left( \frac{3\pi^2 n_i^2}{2m} \right)^{2/3} \frac{K^2}{2m} \propto \rho^{5/3} \text{ if } \rho \text{ constant} \]

Lane Emden Eqn

Jonathan Lane (1872): "an odd-looking odd manoeuvred little man" (Newcomb)

Jacob Emden (1907), "Gebirge" = "Gas balls"

We can combine this polytrope with the hydrostatic and mass continuity equations for a closed set:

\[ \frac{dp}{dr} = -G\rho \frac{dp}{d\rho} \rightarrow \frac{r^2}{\rho} \frac{dp}{d\rho} = -G \rho \]

\[ \rho \frac{d\rho}{dr} - G \frac{dm}{dr} = -4\pi \rho \frac{r^2}{2} \rho \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\rho}{dr} \right) = -4\pi \rho \]

This is Poisson's Eqn: \( \nabla^2 \phi = 4\pi \rho \) for \( \frac{dd}{d\rho} = -\frac{dp}{d\rho} \)

ct. electrosphere: \( \nabla^2 \phi = -4\pi \rho \) for \( -\nabla \phi = \vec{E} \)

Insert \( P = KP^n \)

\[ \Rightarrow \quad K \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho \frac{d\rho}{dr} \right) = -4\pi \rho \]

let \( b = \frac{n+1}{n} \)

\[ \Rightarrow \quad \frac{1}{\rho} \frac{d}{dr} \left( r^2 \rho \frac{d\rho}{dr} \right) = -4\pi \rho \]
It is now useful to change to normalized quantities.

\[ p(r) \equiv \frac{\rho_c}{n} (\text{On}_n(r))^n \quad \rho_c = \text{core density} \]
\[ 0 \leq D_n \leq 1 \]

\[ \Rightarrow \frac{\rho_c}{\rho} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{\text{On}_n^\frac{1}{n}}{n} \rho_c \text{On}_n^{1-\frac{1}{n}} \frac{d\text{On}_n}{dr} \right) = -4\pi \rho_c \text{On}_n^n \]

\[ \Rightarrow \left[ \frac{(n+1) \rho_c}{\text{On}_n^n} \right] \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\text{On}_n}{dr} \right) = -\text{On}_n^n \]

\[ \Rightarrow \alpha^2 = \text{On}_n^n \quad (\alpha^2 \text{ is also } \frac{(n+1) \rho_c}{\text{On}_n^n}) \]

Now introduce a normalized coordinate:

\[ r = \alpha \eta \]

Then

\[ \frac{1}{\alpha^2} \frac{d}{d\eta} \left( \alpha^2 \frac{d\text{On}_n}{d\eta} \right) = -\text{On}_n^n \]

This is the Lane-Emden Equation

\[ \left\{ \begin{array}{l}
\rho(r) = \frac{\rho_c}{n} (\text{On}_n(\alpha \eta))^n \\
\rho(r) = \frac{K \rho_c}{n} (\text{On}_n(\alpha \eta))^{n+1}
\end{array} \right. \]

There are a family of solutions to this equation, but only 3 analytic ones.

\[ D_0 = 1 - \frac{\eta^2}{6} \]
\[ D_1 = \frac{5 \eta^2}{6} \]
\[ D_\infty = \left(1 + \frac{5}{3}\right)^{-\frac{1}{2}} \]

Diagram:

- \( \eta = 0 \)
- \( \eta = 1 \)
- \( \eta = \infty \)

Note:

- \( \eta = 0 \) \( \Rightarrow \) \( \rho_c = \text{core density} \)
- \( \eta = \frac{5}{3} \) \( \Rightarrow \) \( n = 1.5 \)
- \( \eta = \infty \) \( \Rightarrow \) \( n = \frac{3}{2} \)
- \( n = 3, 2 \) \( \Rightarrow \) \( \eta = \frac{\sqrt{6}}{2} \)
- \( n = \frac{4}{3} \) \( \Rightarrow \) \( \eta = \frac{\sqrt{5}}{2} \)
- \( n = \frac{3}{2} \) \( \Rightarrow \) \( \eta = \frac{\sqrt{3}}{2} \)
Boundary conditions (case 2) may be defined as follows:

\[ \text{Surface: } \sigma = 3 \Rightarrow \Delta n = 0 \Rightarrow p = 0 \]
\[ \text{conv: } \sigma = 0 \Rightarrow \frac{\partial p}{\partial r} = 0 \Rightarrow \frac{d\sigma}{dg} |_{\sigma = 0} = 0 \]

Physical parameters also fall out nicely:

\[ R^2 \Delta n_1 (\text{radius}) \]
\[ M \int_0^R \rho(r) r^2 dr = 4\pi R^3 \rho_c \left( \int_0^{\sigma_1} \frac{\Delta n}{d\sigma} d\sigma \right) \]
\[ \frac{d}{dg} \left( \sigma^2 \frac{d\sigma}{dg} \right) = -\sigma^2 D_n \]
\[ M = -4\pi R^2 \rho_c \left( \frac{d\sigma}{dg} \right) |_{\sigma_1} \]

\[ \langle p \rangle = \frac{3}{4\pi} \frac{M}{R^2} = -3\rho_c \left( \frac{d\sigma}{dg} \right) |_{\sigma_1} \Rightarrow \langle p \rangle \propto \rho_c \]

\[ \rho_c = \frac{4\pi G \Delta n \rho_c^2}{n+1}, \quad \frac{4\pi G}{\Delta n} \left[ \pi (n+1) \left( \frac{d\sigma}{dg} \right) |_{\sigma_1} \right]^{-1} \]

What is \( D_n \)? Consider ideal gas:

\[ P = \frac{\mu m}{\mu m} \Rightarrow \rho_c D_n^{n+1} = \frac{\mu m}{\mu m} \rho_c D_n^{n+1} T \]
\[ \Rightarrow D_n = \frac{\mu m}{\mu m} \rho_c T = \frac{T}{T_c} \quad \text{(normalized to core)} \]

The curve \( D_n \) vs. \( \sigma \) therefore provides a rough measure of the temperature gradient in the star - we will approach that separably next.